Asymptotics of discrete spectrum of periodic Schrödinger operator perturbed by non-negative potential and related estimates for singular values of integral operators

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Introduction

Let A be a Schrödinger operator with non-constant metric in $L_2(\mathbb{R}^d)$, $d \geqslant 1$, with periodic coefficients. Let $V(x)$ be a nonnegative potential

$$0 \leqslant V(x) \in L_\infty(\mathbb{R}^d), \quad V(x) \to 0, \quad |x| \to +\infty. \tag{0.1}$$

Consider the perturbed operator

$$B(t) := A + tV, \quad t > 0.$$

We assume

$$\sigma(A) \cap (\alpha, \beta) = \varnothing, \quad \inf \sigma(A) < \alpha < \beta. \tag{0.2}$$

If the conditions (0.1), (0.2) *hold, then*

$$\sigma(B(t)) \cap (\alpha, \beta) \subset \sigma_{discrete}(B(t)), \quad t > 0.$$

As t grows, the eigenvalues of the operator $B(t)$ in the gap (α, β) move from the left to the right; they "are born" at the left edge and they "disappear" at the right edge of the gap.

Introduction

Let's consider the *counting function*

$$N(\lambda,\tau) := \#\{\lambda_j(B(t)) = \lambda, \quad t \in (0,\tau)\}, \quad \lambda \in [\alpha,\beta], \quad \tau > 0.$$

Briefly speaking, the main result is the following: the counting function has the power-like asymptotics at infinity in the case when the perturbation $V(x)$ has a power-like asymptotics at infinity.
If $V(x)$ *has the asymptotics*

$$V(x) \sim \vartheta(x/|x|)|x|^{-\varrho}, \quad |x| \to +\infty, \quad \varrho > 0,$$

then

$$N(\lambda,\tau) \sim \Gamma_\varrho(\lambda)\tau^{d/\varrho}, \quad \tau \to +\infty, \quad \lambda \in (\alpha,\beta). \tag{0.3}$$

The coefficient $\Gamma_\varrho(\lambda)$ *can be calculated in terms of the band functions of the operator* A *and the perturbation* V*. Under certain conditions the asymptotics* (0.3) *holds for* $\lambda = \alpha$*.*

Introduction

Let's consider the *sandwiched resolvent*

$$Z(\lambda) := V^{1/2}(\lambda I - A)^{-1}V^{1/2}, \quad \lambda \in (\alpha, \beta).$$

The operator $Z(\lambda)$, $\lambda \in (\alpha, \beta)$, is self-adjoint and compact. Let's define *the distribution function* of positive eigenvalues of $Z(\lambda)$

$$n_+(s, Z(\lambda)) := \#\{\lambda_j(Z(\lambda)) > s\}, \quad s > 0, \quad \lambda \in (\alpha, \beta).$$

Proposition2.0 (Birman-Schwinger principle)

$N(\lambda, \tau) = n_+(\tau^{-1}, Z(\lambda)), \quad \tau > 0, \quad \lambda \in (\alpha, \beta).$

The proof of asymptotics (0.3) is reduced to the analysis of asymptotics of the positive spectrum for the compact operator $Z(\lambda)$, $\lambda \in (\alpha, \beta)$. Main tools will be the different estimates and asymptotics of the singular number for the various integral operators.

Let a be a $(d \times d)$-matrix-valued function with real entries, $a = a^t$, let b be a real-valued potential

$$a, b \in L_\infty(\mathbb{R}^d), \quad a(x) \geqslant c_0 \mathbf{1}, \quad c_0 > 0, \tag{1.1}$$

$$a(x+n) = a(x), \quad b(x+n) = b(x), \quad n \in \mathbb{Z}^d. \tag{1.2}$$

Consider the operator

$$A = -\mathrm{div} a(x) \mathrm{grad} + b(x),$$

acting in $L_2(\mathbb{R}^d)$, $d \geqslant 1$. The operator A is a selfadjoint operator defined in terms of the corresponding quadratic form.

Let $E_s(k)$, $k \in \tilde{\Omega} := (-\pi, \pi]^d$, $s \in \mathbb{N}$, be the *band* functions of the operator A; $\psi_s(k, x)$, $k \in \tilde{\Omega}$, $x \in \mathbb{R}^d$, $s \in \mathbb{N}$, are the corresponding *Bloch solutions*: $\psi_s(k, \cdot) \in W^1_{2,loc}(\mathbb{R}^d)$,

$$\begin{cases} (-\mathrm{div} a(x)\mathrm{grad} + b(x))\psi_s(k, x) = E_s(k)\psi_s(k, x), & x \in \mathbb{R}^d, \\ \psi_s(k, x + n) = e^{ikn}\psi_s(k, x), & x \in \mathbb{R}^d, \quad n \in \mathbb{Z}^d. \end{cases}$$

Consider the partial isometries from $L_2(\mathbb{R}^d)$ to $L_2(\tilde{\Omega})$

$$\Phi_s u(k) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \overline{\psi_s(k, x)} u(x) dx, \quad u \in L_1(\mathbb{R}^d) \cap L_2(\mathbb{R}^d), \quad s \in \mathbb{N}.$$

The operator A admits the “Floquet decomposition”
$A = \sum_{s \in \mathbb{N}} \Phi_s^* [E_s(k)] \Phi_s$, where $[E_s(k)]$ denotes the multiplication in $L_2(\tilde{\Omega})$ by the function $E_s(k)$. The spectrum $\sigma(A)$ consists of the sequence of *bands* – images of the continuous mappings E_s: $\sigma(A) = \bigcup_{s=1}^{\infty} \mathrm{Im} E_s$.

It is assumed that (α, β) is a gap in $\sigma(A)$ satisfying (for some $l \geqslant 2$) the condition

$$\alpha = \max E_{l-1}(k) < \beta = \min E_l(k), \quad k \in \tilde{\Omega}. \tag{1.3}$$

The operator A is perturbed by the potential V:

$$0 \leqslant V \in L_\infty(\mathbb{R}^d), \quad V(x) \sim \vartheta(x/|x|)|x|^{-\varrho}, \quad |x| \to +\infty, \quad \varrho > 0. \tag{1.4}$$

Consider

$$\Gamma_\varrho(\lambda) := (2\pi)^{-d} d^{-1} \sum_{s=1}^{l-1} \int_{\tilde{\Omega}} (\lambda - E_s(k))^{-d/\varrho}\, dk \int_{\mathbb{S}^{d-1}} \vartheta^{d/\varrho}(\theta) dS(\theta), \qquad \lambda \in [\alpha, \beta].$$

It is clear that $\Gamma_\varrho(\lambda)$ is finite for $\lambda \in (\alpha, \beta)$. The $\Gamma_\varrho(\alpha)$ is finite if

$$(\alpha - E_{l-1})^{-1} \in L_\sigma(\tilde{\Omega}), \quad \sigma = d/\varrho,\ \varrho \in (0, d);\ \sigma = 1,\ \varrho > d;\ \sigma > 1,\ \varrho = d. \tag{1.5}$$

Main result

$$a, b \in L_\infty(\mathbb{R}^d), \quad a(x) \geqslant c_0 \mathbf{1}, \quad c_0 > 0, \tag{1.1}$$

$$a(x+n) = a(x), \quad b(x+n) = b(x), \quad n \in \mathbb{Z}^d. \tag{1.2}$$

$$\alpha = \max E_{l-1}(k) < \beta = \min E_l(k). \tag{1.3}$$

$$0 \leqslant V \in L_\infty(\mathbb{R}^d), \quad V(x) \sim \vartheta(x/|x|)|x|^{-\varrho}, \quad |x| \to +\infty, \quad \varrho > 0. \tag{1.4}$$

$$(\alpha - E_s(\cdot))^{-1} \in L_\sigma(\tilde{\Omega}), \quad s = 1, \dots, l-1. \tag{1.5}$$

Theorem1.1

Assume that $(1.1) – (1.4)$ *are satisfied. Then for all* $\lambda \in (\alpha, \beta)$ *we have*

$$\lim_{\tau \to +\infty} \tau^{-d/\varrho} N(\lambda, \tau) = \Gamma_\varrho(\lambda). \tag{1.6}$$

Under the additional restriction (1.5) *the asymptotics* (1.6) *holds for* $\lambda = \alpha$ *as well.*

Comments

1) Theorem 1.1 has been established in the case $\lambda \in (\alpha, \beta)$, $a(x) = \mathbf{1}$ in the pioneering paper [1] (S.Alama, P.A.Deift, R.Hempel, 1989). In [1] the potential b was not assumed to be periodic. It was only assumed that the so-called integrated density of states exists for the operator $A = -\Delta + b$.

2) The case $\lambda = \alpha$ ($a(x) = \mathbf{1}$, $b(x)$ – periodic) is discussed in the paper [2] (M. Sh. Birman, 1994).

3) Theorem 1.1 has been formulated in the general case in the paper [3] (M. Sh. Birman, 1996) (without proof).

4) Theorem 1.1 has been proved under assumptions $\varrho \in (0, d)$, $d \geqslant 1$ in the paper [4] (M. Sh. Birman, V. S., 2010).

5) Theorem 1.1 has been proved under assumptions $\varrho \geqslant d$, $d \geqslant 1$ in the paper [5] (V. S., 2015).

Comments

6) We do not impose any additional restrictions on the smoothness of the coefficients of the operator A in Theorem 1.1.

7) The order of asymptotics (1.6) differs from the "standard" order $\tau^{d/2}$. The asymptotic coefficient depends on the observation point λ. However, it becomes clear that this asymptotics is of the "Weyl" nature, if the roles of coordinates and (quasi)momentums change.

Let $(\mathcal{Z}, d\nu)$ be a separable measurable space with σ-finite measure. For ν-measurable function $f : \mathcal{Z} \to \mathbb{C}$, we put

$$O_f(s) := \{z \in \mathcal{Z} : |f(z)| > s\}, \quad \nu_f(s) := \nu(O_f(s)), \quad s > 0.$$

With the standard $L_p(\mathcal{Z}, d\nu)$ class let's define

$$L_{p,\infty}(\mathcal{Z}, d\nu) := \{f : \|f\|_{L_{p,\infty}} := \sup_{s>0} s\nu_f^{1/p}(s) < +\infty\}, \quad p > 0.$$

The space $L_{p,\infty}$ is complete with respect to the quasinorm $\|\cdot\|_{L_{p,\infty}}$.

Remark

1. If $f(x) := \omega(x/|x|)|x|^{-d/p}$, $x \in \mathbb{R}^d$, $\omega \in L_p(\mathbb{S}^{d-1})$, then $f \in L_{p,\infty}$, $\|f\|_{L_{p,\infty}} = d^{1/p}\|\omega\|_{L_p}$.
2. If $f \in L_\infty(\mathbb{R}^d)$ and $f(x) = O(|x|^{-d/p})$, $|x| \to \infty$, then $f \in L_{p,\infty}$.

Let T be a compact operator, $\{s_n(T)\}_{n\in\mathbb{N}}$ be singular numbers of T. We define the counting function of singular numbers

$$n(s,T) := \#\{s_n(T) > s\}, \quad s > 0.$$

Let's define the class of compact operators

$$\mathfrak{S}_{p,\infty} := \{T : \|T\|_{\mathfrak{S}_{p,\infty}} := \sup_{s>0} s n^{1/p}(s,T) < \infty\}.$$

The space $\mathfrak{S}_{p,\infty}$ is complete with respect to the quasinorm $\|\cdot\|_{\mathfrak{S}_{p,\infty}}$ and nonseparable. We consider the separable subspace of $\mathfrak{S}_{p,\infty}$

$$\mathfrak{S}^0_{p,\infty} := \{T \in \mathfrak{S}_{p,\infty} : n(s,T) = o(s^{-p}), \quad s \to 0\}.$$

Remark

1. $T \in \mathfrak{S}_{p,\infty} \Leftrightarrow s_n(T) = O(n^{-1/p})$, $\|T\|_{\mathfrak{S}_{p,\infty}} = \sup_{n\in\mathbb{N}} n^{1/p} s_n(T)$.
2. $T \in \mathfrak{S}^0_{p,\infty} \Leftrightarrow s_n(T) = o(n^{-1/p})$.

For compact self-adjoint operator T we consider the counting functions for negative and positive spectrum:

$$n_\pm(s,T) = \#\{\pm\lambda_j(T) > s\}, \quad s > 0.$$

It's clear that

$$n(s,T) = n_+(s,T) + n_-(s,T), \quad T = T^*; \quad n(s,T) = n_+(s,T), \quad T \geqslant 0.$$

Let's define the following continuous functionals on $\mathfrak{S}_{p,\infty}$:

$$\Delta_p(T) := \limsup_{s\to+0} s^p n(s,T), \quad \delta_p(T) := \liminf_{s\to+0} s^p n(s,T), \quad T \in \mathfrak{S}_{p,\infty};$$

$$\Delta_p^\pm(T) := \limsup_{s\to+0} s^p n_\pm(s,T), \quad \delta_p^\pm(T) := \liminf_{s\to+0} s^p n_\pm(s,T),$$
$$T = T^* \in \mathfrak{S}_{p,\infty}.$$

Remark

1. $D_p(T+K) = D_p(T)$, $D = \Delta, \delta, \Delta^{\pm}, \delta^{\pm}$, $T \in \mathfrak{S}_{p,\infty}$, $K \in \mathfrak{S}^0_{p,\infty}$.
2. $\|T_1 T_2\|_{\mathfrak{S}_{p,\infty}} \leqslant C(p_1, p_2) \|T_1\|_{\mathfrak{S}_{p_1,\infty}} \|T_2\|_{\mathfrak{S}_{p_2,\infty}}$, $p^{-1} = p_1^{-1} + p_2^{-1}$.
3. $\|ATB\|_{\mathfrak{S}_{p,\infty}} \leqslant \|A\| \|T\|_{\mathfrak{S}_{p,\infty}} \|B\|$.

Let $(\mathcal{X}, d\varrho)$, $(\mathcal{Y}, d\tau)$ be two separable spaces with σ-finite measures. Let $T : L_2(\mathcal{Y}, d\tau) \to L_2(\mathcal{X}, d\varrho)$ be a linear bounded integral operator with the kernel $t(\cdot,\cdot) \in L_\infty(\mathcal{X} \times \mathcal{Y}, d\varrho d\tau)$. Let $f : \mathcal{X} \to \mathbb{C}$ and $g : \mathcal{Y} \to \mathbb{C}$ be measurable functions. We consider an integral operator $fTg : L_2(\mathcal{Y}, d\tau) \to L_2(\mathcal{X}, d\varrho)$ with the kernel $f(x)t(x,y)g(y)$.

Proposition2.1 ([6], M.Sh.Birman, M.Z.Solomyak, 1990)

If $f \in L_{p,\infty}(\mathcal{X}, d\varrho)$, $g \in L_p(\mathcal{Y}, d\tau)$, $p > 2$, *then* $fTg \in \mathfrak{S}_{p,\infty}$,

$$\|fTg\|_{\mathfrak{S}_{p,\infty}} \leqslant C(p)\|T\|^{1-2/p}\|t(\cdot,\cdot)\|_{L_\infty}^{2/p}\|f\|_{L_{p,\infty}}\|g\|_{L_p}.$$

If, in addition, $\varrho_f(s) = o(s^{-p})$, $s \to +0$, *then* $fTg \in \mathfrak{S}^0_{p,\infty}$.

Let $(\mathcal{X}, d\varrho)$, $(\mathcal{Y}, d\tau)$ be two separable spaces with σ-finite measures. Let $T : L_2(\mathcal{Y}, d\tau) \to L_2(\mathcal{X}, d\varrho)$ be a linear bounded integral operator with the kernel $t(x, y)$, $x \in \mathcal{X}, y \in \mathcal{Y}$. Let $f : \mathcal{X} \to \mathbb{C}$ and $g : \mathcal{Y} \to \mathbb{C}$ be measurable functions. We consider an integral operator $fTg : L_2(\mathcal{Y}, d\tau) \to L_2(\mathcal{X}, d\varrho)$ with the kernel $f(x)t(x, y)g(y)$. We give the estimates for singular numbers in terms of the function $(fg)(x, y) := f(x)g(y)$ and the measure $d\nu(x, y) := |t(x, y)|^2 d\varrho(x)d\tau(y)$.

Proposition2.2 ([7], V.S., 2009)

If $fg \in L_{p,\infty}(\mathcal{X} \times \mathcal{Y}, d\nu)$, $p > 2$, *then* $fTg \in \mathfrak{S}_{p,\infty}$,

$$\|fTg\|_{\mathfrak{S}_{p,\infty}} \leqslant C(p)\|T\|^{1-2/p}\|fg\|_{L_{p,\infty}}.$$

If, in addition, $\nu_{fg}(s) = o(s^{-p})$, $s \to +0$, *then* $fTg \in \mathfrak{S}^0_{p,\infty}$.

Let $A = -\mathrm{div} a(x) \mathrm{grad} + b(x)$ be the operator acting in $L_2(\mathbb{R}^d)$ with the coefficients satisfying (1.1), (1.2). Let $\varphi(\lambda)$, $\lambda \in \mathbb{R}$ be a bounded Borel function; $W \in L_\infty(\mathbb{R}^d)$.

Proposition2.3

- *If* $\varphi(\lambda) = O(\lambda^{-d/2p-\varepsilon})$, $\lambda \to +\infty$, $W(x) = O(|x|^{-d/p})$, $|x| \to \infty$, $p > 0$, $\varepsilon > 0$, *then* $\varphi(A)W \in \mathfrak{S}_{p,\infty}$.
- *If* $\varphi(\lambda) = O(\lambda^{-d/2p-\varepsilon})$, $\lambda \to +\infty$, $W(x) = o(|x|^{-d/p})$, $|x| \to \infty$, $p > 0$, $\varepsilon > 0$, *then* $\varphi(A)W \in \mathfrak{S}^0_{p,\infty}$.

Note that the case $p \in (0, 2]$ can not be treated by the standard Cwikel estimate for the singular numbers.
Proposition 2.3 follows from more general statement.

Let $A = -\mathrm{div} a(x) \mathrm{grad} + b(x)$ be the operator acting in $L_2(\mathbb{R}^d)$ with the coefficients satisfying (1.1), (1.2). Let's denote $a_0 := min\{\inf \sigma(A), 0\}$, $a_j = 2^j$, $j \in \mathbb{N}$. With any bounded Borel function $\varphi(\lambda)$, $\lambda \in \mathbb{R}$, we associate the sequence

$$u(\varphi) := \{u_j(\varphi)\}_{j=0}^{\infty}, \quad u_j(\varphi) := \sup_{\lambda}\{|\varphi(\lambda)|, \lambda \in [a_j, a_{j+1})\}, \quad j \in \mathbb{Z}_+.$$

Putting $\Omega := [0,1)^d$, $\Omega_n := \Omega + n$, $n \in \mathbb{Z}^d$, for each function $W \in L_{2,loc}(\mathbb{R}^d)$ we introduce the sequence

$$v(W) := \{v_n(W)\}_{n \in \mathbb{Z}^d}, \quad v_n(W) := \left(\int_{\Omega_n} |W(x)|^2 dx \right)^{1/2}, \quad n \in \mathbb{Z}^d.$$

Denote by $u(\varphi)v(W)$ the sequence $\{u_j(\varphi)v_n(W)\}_{(j,n) \in \mathbb{Z}_+ \times \mathbb{Z}^d}$; introduce the measure $d\nu := a_{j+1}^{d/2} dj\, dn$ on the set $\mathbb{Z}_+ \times \mathbb{Z}^d$, where dj, dn are the counting measures on $\mathbb{Z}_+$, $\mathbb{Z}^d$.

Proposition2.3* ([8], V.S., 2013)

If $u(\varphi)v(W) \in L_{p,\infty}(\mathbb{Z}_+ \times \mathbb{Z}^d, d\nu)$, $p \in (0,2)$, *then* $\varphi(A)W \in \mathfrak{S}_{p,\infty}$,

$$\|\varphi(A)W\|_{\mathfrak{S}_{p,\infty}} \leqslant C\|u(\varphi)v(W)\|_{L_{p,\infty}}.$$

If $\nu_{uv}(s) = o(s^{-p})$, $s \to +0$, *then* $\varphi(A)W \in \mathfrak{S}^0_{p,\infty}$.

Let $A = -\mathrm{div}\, a(x) \mathrm{grad} + b(x)$ be the operator acting in $L_2(\mathbb{R}^d)$ with the coefficients satisfying (1.1), (1.2); $W \in L_\infty(\mathbb{R}^d)$ has asymptotics

$$W(x) = \omega(x/|x|)|x|^{-d/p} + o(|x|^{-d/p}), \quad |x| \to +\infty, \quad p \in (0, +\infty).$$

Proposition2.4 ([9], V.S., 2014)

If $\varphi(\lambda)$, $\lambda \in \mathbb{R}$, is a continuous compactly supported function, then

$$\varphi(A)W - W\varphi(A) \in \mathfrak{S}^0_{p,\infty},$$
$$D_p(\varphi(A)W) = D_{p/n}(\varphi^n(A)W^n), \quad n \in \mathbb{N}, \quad D = \Delta, \delta.$$

The sketch of the proof for Theorem 1.1

$$N(\lambda,\tau) = n_+(\tau^{-1}, Z(\lambda)), \quad Z(\lambda) := V^{1/2}(\lambda I - A)^{-1}V^{1/2}.$$

Let's denote $E_1 := E_A[\inf\sigma(A), \alpha]$, $E_2 := E_A[\beta, +\infty)$. The inequality

$$Z(\lambda) \leqslant Z_1(\lambda) := V^{1/2}E_1(\lambda I - A)^{-1}E_1V^{1/2}$$

implies $N(\lambda,\tau) \leqslant n_+(\tau^{-1}, Z_1(\lambda))$.

On the other hand, from the variational considerations we obtain

$$N(\lambda,\tau) = n_+(\tau^{-1}, Z(\lambda)) \geqslant n_+(\tau^{-1}, E_1Z(\lambda)E_1).$$

It follows from Propositions 2.3 and 2.4 that $Z_1(\lambda), E_1Z(\lambda)E_1 \in \mathfrak{S}_{d/\varrho,\infty}$ and $E_1Z(\lambda)E_1 - Z_1(\lambda) \in \mathfrak{S}^0_{d/\varrho,\infty}$, because $V(x)$ has the asymptotics $V(x) \sim \vartheta(x/|x|)|x|^{-\varrho}, \quad |x| \to +\infty$. Hence,

$$\delta_{d/\varrho}(Z_1(\lambda)) \leqslant \liminf_{\tau\to+\infty} \tau^{d/\varrho}N(\lambda,\tau) \leqslant \limsup_{\tau\to+\infty} \tau^{d/\varrho}N(\lambda,\tau) \leqslant \Delta_{d/\varrho}(Z_1(\lambda)).$$

The sketch of the proof for Theorem 1.1

The proof of asymptotics (1.6) is reduced to the analysis of asymptotics for singular numbers of the operator

$$Z_1(\lambda) := V^{1/2}E_1(\lambda I - A)^{-1}E_1 V^{1/2} = \sum_{s=1}^{l-1} V^{1/2}\Phi_s^*[(\lambda - E_s(k))^{-1}]\Phi_s V^{1/2}.$$

If $\varrho \in (0, d)$, the study of the asymptotics for the singular numbers of the operator $Z_1(\lambda)$ can be simply reduced to investigation of asymptotics for the singular numbers of integral operators acting in $L_2(\tilde{\Omega})$

$$X_{s,t} := [f(k)]\Phi_s V\Phi_t^*[h(k)], \quad s, t \in \mathbb{N}.$$

The sketch of the proof for Theorem 1.1

Proposition3.1

Suppose that the potential V has the asymptotics $V(x) \sim \vartheta(x/|x|)|x|^{-\varrho}, \quad |x| \to +\infty$, and let $f, h \in L_{2d/\varrho}(\tilde{\Omega})$, $\varrho \in (0, d)$. Then the distribution function of singular numbers of the operator $X_{s,t}$ has the asymptotics

$$\lim_{\sigma \to +0} \sigma^{d/\varrho} n(\sigma, X_{s,t}) = d^{-1}(2\pi)^{-d} \delta_{st} \int_{\tilde{\Omega}} |f(k)h(k)|^{d/\varrho} dk \times \times \int_{\mathbb{S}^{d-1}} |\omega(\theta)|^{d/\varrho} dS(\theta). \quad (2.1)$$

To prove Proposition 3.1 we need to expand the kernels of the operators Φ_s, Φ_t in a Fourier series: $\psi_s(k, x) = e^{ikx} \sum_{n \in \mathbb{Z}^d} c_n^s(k) e^{2\pi inx}$. For the finite part of Fourier series the formula (2.1) is more or less obvious. The limiting procedure is confirmed by Proposition 2.2.

The sketch of the proof for Theorem 1.1

For $\varrho \in (0, d)$ Proposition 3.1 yields

$$\delta_{d/\varrho}(Z_1(\lambda)) = \Delta_{d/\varrho}(Z_1(\lambda)) = \Gamma_\varrho(\lambda). \tag{2.2}$$

For $\varrho \geqslant d$, (2.2) can be justified by Proposition 2.4.

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