Complex WKB method for difference equations in unbounded domains

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$$\psi(z+h)+\psi(z-h)+v(z)\psi(z)=E\psi(z), \quad z\in\mathbb{C},$$

$h>0$ and $E\in\mathbb{C}$ are parameters, v is entire.

The problem is to get asymptotics of ψ as $h\to 0$.

$$\left(2\cos\frac{h}{i}\frac{d}{dz}+v(z)\right)\psi(z)=E\psi(z), \quad z\in\mathbb{C},$$

Quasiclassical limit $h\to 0$
for ODE: **the classical complex WKB method**;
for **the Harper equation** ($v(z)=2\cos z$): V. Buslaev, A. Fedotov;
for $v(z)=\sum_{k=-m}^{n} c_k e^{ikz}$ we use new ideas (A. Fedotov, F. Klopp).

The classical complex WKB method for ODE

$$-h^2\psi''(z) + v(z)\psi(z) = E\psi(z), \quad z \in \mathbb{C},$$

- complex momentum p: $p^2(z) + v(z) = E$;
- action θ: $\theta(z) = \int_{z_0}^{z} p\,dz$;
- regular set: no branch points of p;
- canonical curve γ: $\mathrm{Im}\ \theta$ is monotonically increasing along γ;
- canonical domain K: $\forall z \in K$ there exists a canonical curve $\gamma \subset K$ connecting z to a fixed point $z_* \in \overline{K}$.

The classical complex WKB method for ODE

Theorem Let K be a canonical domain. Then, for sufficiently small h, there exist $\psi_\pm$, two entire solutions to

$$-h^2\psi''(z) + v(z)\psi(z) = E\psi(z), \quad z \in \mathbb{C},$$

having in K the asymptotic representations

$$\psi_\pm(z) = \frac{e^{\pm\frac{i}{h}\theta(z)+O(h)}}{\sqrt{p(z)}}, \quad \theta(z) = \int_{z_0}^{z} p\,dz, \quad h \to 0.$$

The error estimate is locally uniform in $z \in K$.

$$\left(2\cos\frac{h}{i}\frac{d}{dz}+v(z)\right)\psi(z)=E\psi(z), \quad z\in\mathbb{C},$$

- complex momentum p: $2\cos p(z)+v(z)=E$;
- vertical curve γ: angles between γ and $\mathrm{Im} z=C$ are non-zero;
- two actions θ, θ_π: $\theta(z)=\int_{z_0}^{z} p\,dz,\ \theta_\pi=\int_{z_0}^{z}(p-\pi)\,dz$;
- canonical curve γ: $\mathrm{Im}\ \theta$ is monotonically increasing and $\mathrm{Im}\ \theta_\pi$ is monotonically decreasing along γ;
- horizontally connected domain D: $[z_1, z_2]\subset D$, if $\mathrm{Im} z_1=\mathrm{Im} z_2$;
- canonical domain K: K is a union of canonical curves going from $-i\infty$ to $+i\infty$.

The main theorem

Let $v(z) = \sum_{k=-m}^{n} c_k e^{ikz}, m, n > 0, c_n, c_{-m} \neq 0$ and K be a canonical domain. Then, for sufficiently small h, there exist $\psi_\pm$, two entire solutions to

$$\psi(z+h) + \psi(z-h) + v(z)\psi(z) = E\psi(z), \quad z \in \mathbb{C},$$

having in K the asymptotic representations

$$\psi_\pm(z) = \frac{e^{\pm \frac{i}{h}\theta(z) + O(h)}}{\sqrt{\sin p(z)}}, \quad \theta(z) = \int_{z_0}^{z} p\, dz, \quad h \to 0.$$

The error estimate is uniform in $K_\epsilon = \{z \in K \mid \mathrm{dist}(z, \partial K) > \epsilon\}$ $\forall \epsilon > 0$.

Plan of the proof

$$\psi(z+h) + \psi(z-h) + v(z)\psi(z) = E\psi(z), \quad z \in K.$$

- **Matrix equation**:

  (1) $\Psi(z+h) = M(z)\Psi(z), \quad M(z) = \begin{pmatrix} E - v(z) & -1 \\ 1 & 0 \end{pmatrix};$

- **"Diagonalization"**: $\Psi(z) = U(z)\Phi(z)$,

  $$D(z) = U^{-1}(z)M(z)U(z) = \begin{pmatrix} e^{ip(z)} & 0 \\ 0 & e^{-ip(z)} \end{pmatrix},$$

  (2) $\Phi(z+h) = T(z)\Phi(z), \quad T(z) = D(z) + O(h)$ as $h \to 0$;

Plan of the proof

- **"Variation of parameters"**:

$$\Phi(z) = \begin{pmatrix} e^{\frac{i}{h}\theta(z)+O(1)} & 0 \\ 0 & e^{-\frac{i}{h}\theta(z)+O(1)} \end{pmatrix} X(z), \quad \theta(z) = \int_{z_0}^{z} p\, dz,$$

(3) $X(z+h) - X(z) = S(z)X(z),$

$$S(z) = \begin{pmatrix} 0 & O\left(he^{-\frac{2i}{h}\theta(z)}\right) \\ O\left(he^{\frac{2i}{h}\theta(z)}\right) & 0 \end{pmatrix};$$

Invert the first-order difference operator in the left-hand side of (3).

First-order difference equation

$$g(z+h) - g(z) = f(z) \qquad (1)$$

Lemma Let $D \subset \mathbb{C}$ be a horizontally connected domain that consists of vertical curves going from $-i\infty$ to $+i\infty$, and let f be an analytic function in D, $f(z) = O\left(\frac{1}{z^2}\right)$ as $|\mathrm{Im}\, z| \to \infty$. By $\gamma_z \subset D$ we denote a vertical curve going from $-i\infty$ to $+i\infty$ via z. Then

$$g(z) = \mathcal{L}f\,(z) := \frac{1}{2ih} \int\limits_{\gamma_z} \cot\left[\frac{\pi(\zeta - z - 0)}{h}\right] f\,(\zeta)\, d\zeta$$

is an analytic solution to (1) in $\{z \in \mathbb{C} \mid z, z+h \in D\}$.

Singular integral equation

(4) $Y(z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \hat{K} Y(z), \; \hat{K} = \begin{pmatrix} 0 & \mathcal{L} O\left(h e^{-\frac{2i}{h}\theta}\right) \\ \mathcal{L} O\left(h e^{\frac{2i}{h}\theta}\right) & 0 \end{pmatrix},$

$$\mathcal{L} f(z) = \frac{1}{2ih} \int\limits_{\gamma_z} \left(\cot\left[\frac{\pi(\zeta - z - 0)}{h}\right] - i \right) f(\zeta)\, d\zeta,$$

(5) $Y_1 = 1 + \mathcal{L} O(h) \mathcal{K} O(h) Y_1,$

$$\mathcal{K} f(z) = e^{-\frac{2i}{h}\theta(z)} \mathcal{L}\left(e^{\frac{2i}{h}\theta} f\right)(z).$$

γ_z is **canonical** $\Rightarrow \|\mathcal{L} O(h) \mathcal{K} O(h)\|$ is **small**.