Abstracts of Lectures and Talks

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Lectures

Larissa V. Fardigola
Transformation operators in control problems

In [3], a novel transformation operator $\widetilde{\mathbb{T}}$ is introduced and studied for the differential operator $\dfrac{1}{\rho(x)}\dfrac{d}{dx}\left(k(x)\dfrac{d}{dx}(\cdot)\right)$. Here $\rho \in C^1[0,+\infty)$ and $k \in C^1[0,+\infty)$ are positive functions on $[0,+\infty)$, and they satisfy some additional restrictions. Together with this operator we introduce the special modified spaces $\mathbb{H}^m$, $m = \overline{-2,2}$, of the Sobolev type where the space $L^2(\mathbb{R})$ is replaced by the space $L^2_\rho(\mathbb{R})$ with the weight $\sqrt{\widehat{\rho}}$ and the differential operator d/dx is replaced by the "linearly deformed" one $\sqrt{\widehat{k}/\widehat{\rho}}\Big(d/dx + \big(\widehat{\rho}'/\widehat{\rho} + \widehat{k}'/\widehat{k}\big)/4\Big)$, $\widehat{k}$ and $\widehat{\rho}$ are the even extensions of k and ρ, respectively. The growth of distributions from these spaces is associated with the differential operator data ρ and k. Some extension of the classical transformation operator for the Sturm–Liouvolle problem saving the asymptotics of solutions at infinity [2] is also used in the talk.

Together with the transformations operators, the influence operator Ψ and its modification $\widehat{\Psi}$ are introduced and studied [1]. These operators are non-bijective analogs of transformation operators. They are used for studying the wave equation $w_{tt} = w_{xx} - q^2 w$, $x > 0$, $t \in (0,T)$, controlled by the Dirichlet boundary condition: $w(0,t) = u(t)$, where $q \geq 0$, $T > 0$ are constants, $u \in L^\infty(0,T)$ is a control. This control system is considered in classical Sobolev spaces H^s . By using the influence operators Ψ and $\widehat{\Psi}$, necessary and sufficient conditions for L^∞-controllability and approximate L^∞-controllability are obtained at a given time $T > 0$ and at a free time. It is proved that controllability properties are similar at a given time in the cases $q = 0$ and $q > 0$. It is also proved that the case $q = 0$ essentially differs from the case $q > 0$ at a free time. In particular, if $q > 0$, then each initial state is approximately L^∞-controllable at a free time. However, if $q = 0$, then an initial state (w, w_t) of this system is approximately L^∞-controllable at a free time iff $w_t(\cdot,0) = w_x(\cdot,0)$. A similar relation is necessary for L^∞-controllability and approximate L^∞-controllability at a given time in the both cases: $q = 0$ and $q > 0$.

In [3], the wave equation $z_{tt} = \frac{1}{\rho}\left(k z_x\right)_x + \gamma z$, $x > 0$, $t \in (0,T)$, is studied in the modified spaces $\mathbb{H}^m$ of the Sobolev type. Here ρ, k, and γ are given functions on $[0,+\infty)$ under some restrictions. By using the transformation operator $\widetilde{\mathbb{T}}$ introduced and studied here, we see that this equation controlled by the Dirichlet boundary condition replicates the controllability properties of the auxiliary wave equation $w_{tt} = w_{xx} - q^2 w$, $x > 0$, $t \in (0,T)$, controlled by the boundary condition of the same type. Here $q \geq 0$ is a constant determined by ρ, k, and γ. The auxiliary equation is considered in the classical Sobolev spaces. Thus, necessary and sufficient conditions

of L^∞-controllability and approximate L^∞-controllability at a given time $T > 0$ and at a free time are obtained for the main system from those for the auxiliary system.
In fact, all the main results obtained in the talk for the wave equation with constant coefficients are generalized to the case of the wave equation with variable coefficients using the transformation operator $\widetilde{\mathbb{T}}$.

References

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Alexander Fedotov
Monodromization Method in the Theory of Quasi-periodic Equations

The lectures are devoted to the monodromization method, a renormalization approach proposed by V. Buslaev and A. Fedotov to study difference equations with periodic coefficients on the complex plane. It arose when they tried to generalize ideas of Bloch-Floquet. Later, Alexander Fedotov and Frédéric Klopp have understood that the monodromization method is a natural tool to study one dimensional quasi-periodic differential and difference equations. For main ideas, constructions and results of the method see review [1], paper [2], and the references quoted therein.
In the course we first recall basic ideas and constructions of the method. Then, we describe how the method is used to analyze the Cantorian structure of the spectrum of the Harper operator

$$H_h\psi(x) = \psi(x+h) + \psi(x-h) + 2\cos x\,\psi(x) \quad \text{in} \quad L^2(\mathbb{R})$$

in the case when the translation parameter h is represented by a quasi-classical continuous fraction, i.e. a continuous fraction with sufficiently large elements (results of V. Buslaev and A. Fedotov). Next, we describe applications of the method to the study of two-frequency quasi-periodic differential equations (results of A. Fedotov and F. Klopp). And finally, we discuss how this method is used in the theory of quasi-periodic difference equations on the integer lattice.

References

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Nikolai Filonov
Spectral Theory of the Maxwell Operator

The Maxwell operator appears in many different problems related to the electromagnetic waves. First of all, one has to find a "correct" realization of it as a self-adjoint operator in a suitable Hilbert space. In the 1st lecture, we give such a definition of the Maxwell operator, which works also in the case of nonsmooth boundary and/or nonsmooth coefficients describing the medium filling the domain (dielectric and magnetic permittivities). Next, we give an analytic description of the functions belonging to the domain of the operator. If the domain is convex, then the "weak" operator coincides with the "strong" one [FP]. In the more general case of Lipschitz domain we describe the possible singularities of the electromagnetic fields [BS87].
In the bounded domain with Lipschitz boundary the spectrum of the Maxwell operator is discrete, and symmetric with respect to zero. In the 2nd lecture, we discuss the asymptotic behaviour of its eigenvalues. In the smooth case such asymptotics was established by H. Weyl [W]. Finding and justification of the asymptotics in the general case have taken about one hundred years [BF].
In the 3rd lecture, we discuss the properties of the Maxwell operator with periodic coefficients - in the whole space, in a layer, and in a cylinder. The periodic Maxwell operator is used in the theory of photonic crystals. It is well known that the spectrum has a band-gap structure. We will show that in the isotropic case, when the coefficients of the operator are smooth enough scalar functions, the spectrum is absolutely continuous [M], [Su].

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Andrii Khradbustovskyi
Spectral Problems for Elliptic Operators in Strongly Perforated Domains

In this talk we treat the eigenvalue problems

$$(1) \qquad \begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega^\varepsilon, \\ u = 0 & \text{on } \partial\Omega^\varepsilon \end{cases} \qquad \text{(Dirichlet problem)}$$

and

$$(2) \qquad \begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega^\varepsilon, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega^\varepsilon \end{cases} \qquad \text{(Neumann problem)}$$

posed in domains $\Omega^\varepsilon \subset \mathbb{R}^n$ with a very complicated microstructure. The parameter $\varepsilon \ll 1$ is a characteristic scale of this microstructure. We mostly focus on the case, when Ω^ε is obtained from a bounded domain Ω by removing a lot of tiny holes. The holes are distributed periodically in $\mathbb{R}^n$, with the period ε. When $\varepsilon \to 0$ the number of holes in Ω tends to infinity.

Our goal is to describe the asymptotic behaviour of the problems (1) and (2). Namely, we find the *homogenized spectral problem* in Ω, whose spectrum "attracts" the spectrum of (1)/(2) as $\varepsilon \to 0$. The form of the homogenized spectral problem essentially depends on the type of boundary conditions on the holes (Dirichlet/Neumann) and the decay rate of their radii as $\varepsilon \to 0$.

We also discuss the same type problem for domains with more involved microstructure, namely, the so-called "weakly connected domains". In this case the homogenized spectral problem may have peculiar spectral properties (for example, non-empty essential spectrum).

Peter Müller
Anderson Orthogonality

Large Fermion gases change their ground state drastically when subjected to a perturbation, even if the perturbation is small. This phenomenon is called Anderson orthogonality and is considered as a fundamental characteristic of Fermi gases, both non-interacting and interacting. In its simplest version, Anderson orthogonality was mathematically understood in recent years with the help of spectral analysis for Schrödinger operators. The argument relies on certain asymptotics of products and differences of spectral projections to analyse Fredholm determinants. We will explain the basic mathematical arguments and also a relation between Anderson orthogonality and the spectral shift function.

Leonid Pastur
Analogs of the Szegő Theorem for Ergodic Operators

In the first part of the mini-course we consider an asymptotic setting for ergodic operators generalizing that for the Szegő theorem on the determinants of finite-dimensional restrictions of the Toeplitz operators [1]. It is motivated by certain problems of quantum mechanics and quantum informatics [2] and formulated via the asymptotic trace formula determined by a triple consisting of an ergodic operator and two functions, the symbol and the test function. In the frameworks of this setting we analyze two important examples of ergodic operators: the one dimensional discrete Schrödinger operator with random i.i.d. potential and the same operator with quasiperiodic potential. In the random case we find that for smooth symbols the corresponding asymptotic formula contains a new subleading term, which is random and proportional to the square root of the length of the interval of restriction. The origin of the term are the Gaussian fluctuations of the corresponding trace, that is, in fact, the Central Limit Theorem for the trace. We also present an example of a non-smooth symbol for which the subleading term is the sum of two ergodic processes bounded with probability 1, while for the convolution operators and the same symbol the subleading term grows logarithmically in the length of the interval. In the quasiperiodic case and for smooth symbols the subleading term is bounded as in the Szegő theorem but unlike the theorem, where the term does not depend on the length, in the quasiperiodic case the term is the sum of two ergodic processes in the length of the interval of restriction [3]. In the second part of the mini-course we discuss one of the problems of quantum informatics, which can be formulated as a version of the Szegő theorem. We give a brief description of the corresponding notions and basic results on the entanglement entropy of macroscopic [4] systems and then present recent results [5] on the entanglement entropy of the d-dimensional quasifree fermions whose one body Hamiltonian is the discrete Schrodinger operator with random potential. Using basic facts on Anderson localization, we show first that the disorder averaged entanglement entropy $\langle S_L \rangle$ of the d dimension cube of side length L admits the area law asymptotic scaling $\langle S_L \rangle \sim L^{(d-1)}$, $L \gg 1$ even in the gapless case, thereby manifesting the area law in the mean for our model. For $d = 1$ and $L \gg 1$ we obtain then asymptotic bounds for the entanglement entropy of typical realizations of disorder and use them to show that the entanglement entropy is not self-averaging, that is, has non vanishing random fluctuations even if $L \gg 1$.

References

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Mariya Shcherbina
Classical Models of Random Matrix Theory

Random matrices is an active field of research in mathematics and physics. It has became very active since the end of the 1970s under the flow of accelerating impulses from quantum field theory, quantum mechanics (quantum chaos), statistical mechanics, and condensed matter theory in physics, probability theory, statistics, combinatorics, operator theory, number theory, and theoretical computer science in mathematics, and also telecommunication theory, qualitative finances, structural mechanics, etc. In addition to its mathematical richness random matrix theory was successful in describing various phenomena of these fields, providing them with new concepts, techniques, and results (see [1] and references therein).
There are two main directions in studies on the random matrix theory: so-called global and local regimes of the eigenvalue statistics. In both directions there are a lot of important results obtained in the last years, in particular, a number of results on the fluctuations of linear eigenvalue statistics, proving central limit theorems in the case of non analytic test functions [2], [3]. It was established also non Gaussian

behavior of fluctuation in the case of β-matrix models [4]. An essential progress has been achieved in the studies of local regimes of β-matrix models [5].
In the first part of the course it is planned to present the basic results of the random matrix theory and in the second part certain recent results will be discussed.

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Dmitry Shepelsky
Boundary Value Problems for Integrable Nonlinear Equations

Integrable non-linear partial differential equations comprise now a vast and active branch of mathematics and mathematical physics. However, any adaptation of the main method of inverse scattering transforms to the study of initial boundary value (IBV) problems for nonlinear evolution equations possessing a Lax pair representation faces a major problem: the evolution of the spectral data requires the knowledge of an "excessive" amount of boundary values, which cannot all be given as boundary conditions for a well-posed IBV problem. Only for certain particular classes of boundary conditions, the IBV problem remains completely integrable, i.e. solving it reduces to solving a series of well-posed linear problems.
Nevertheless, a general method for analyzing such IBV problem has been recently proposed, see [1], [2], [3], [4], [5], which is based on the simultaneous spectral analysis of the two eigenvalue equations of the associated Lax pair. It allows expressing the solution in terms of the solution of an associated matrix Riemann-Hilbert (RH) problem formulated in the complex plane of the spectral parameter. The spectral

functions determining the RH problem are expressed in terms of the boundary values of the solution. The fact that these boundary values are in general related can be expressed in a simple way in terms of a *global relation* satisfied by the corresponding spectral functions. A thorough analysis of the global relation gives means for obtaining important information about the solution of the IBV problem even in the general (non-integrable) case.
The first part of the course will deal with basic facts of the method of inverse scattering transforms and the Riemann-Hilbert problem, while in the second part recent results will be presented.

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Nataila Smorodina
Limit Theorems for Sums of Independent Random Variables and Feynman Integral

Using classical probabilistic methods we construct a probabilistic approximation in L_2 of the Cauchy problem solution for an equation $\frac{\partial u}{\partial t} = \frac{\sigma^2}{2}\Delta u + V(x)u$, where $\operatorname{Re} V \leqslant 0$ and σ is a complex parameter with $\operatorname{Re}\sigma^2 \geqslant 0$. This equation coincides with the heat equation when $\operatorname{Im}\sigma = 0$ and with the Schrödinger equation when $\operatorname{Re}\sigma^2 = 0$.

Wolfgang Spitzer
Area Law for the Entanglement Entropy of the free Fermi Gas at nonzero Temperature

We consider the entanglement entropy (or mutual information) of thermal equilibrium states of non-interacting fermions in the d-dimensional Euclidean space, $\mathbb{R}^d$. This entropy can be expressed as a trace of a non-smooth function of Wiener–Hopf type operators on the Hilbert space $\mathrm{L}^2(\mathbb{R}^d)$. We first show, for any dimension $d \geq 1$, how to define this entropy properly in terms of the temperature-dependent Fermi symbol, the Rényi entropy function η_γ (with parameter $\gamma > 0$) from $\mathbb{R}$ into $[0, \log(2)]$, and a "localizing" domain $\Lambda \subset \mathbb{R}^d$ (and its complement). Then we determine for dimension $d = 1$ the precise leading term of the asymptotic expansion as Λ grows up to the full real line, $\mathbb{R}$ in agreement with an area law. Here, we basically follow Harold Widom's seminal ideas, who proved such expansions for smooth functions of Wiener–Hopf operators. The necessary extension to non-smooth functions such as η_γ was made possible due to recent advances in semi-classical asymptotic analysis by one of us (AVS). Finally, we consider the (newly found) asymptotic coefficient in the zero-temperature limit $T \downarrow 0$ and found consistency with the corresponding (ground-state) result for $T = 0$. The latter has been proved recently by us.
This is joint work with H. Leschke and A.V. Sobolev.

Tatiana Suslina
Periodic Differential Operators: Threshold Properties and Homogenization

Homogenization theory studies properties of differential operators with periodic rapidly oscillating coefficients. It is a wide area of mathematical research with a vast number of applications in natural sciences and engineering. The basic intuitive idea behind homogenization is that a medium with rapidly oscillating properties behaves like a homogeneous medium with effective properties.
Let A_ε be a self-adjoint elliptic second order operator in $\mathbb{R}^d$ with periodic coefficients depending on x/ε. Here $\varepsilon > 0$ is a small parameter. The simplest example of such operator is $A_\varepsilon = -\mathrm{div} g(x/\varepsilon) \mathrm{grad}$. In operator terms, the basic result of homogenization theory claims that the resolvent of A_ε converges (in some sense) to the resolvent of the effective operator A^0 with constant effective coefficients, as ε tends to zero.
We will discuss the operator-theoretic approach to homogenization problems suggested by Birman and Suslina [1], [2], [3]. This approach allows to prove convergence of the resolvent of A_ε in the L_2-operator norm with sharp order error estimate. Such estimates are called operator error estimates in homogenization theory. Also, we find more accurate approximation for the resolvent with a corrector taken into account.

The method is based on the scaling transformation, the Floquet-Bloch theory and the analytic perturbation theory. We study the spectral properties of the operator A_1 (with $\varepsilon = 1$) near the bottom of its spectrum, i. e., the threshold properties. It turns out that the threshold properties are responsible for homogenization procedure. It means that, from the spectral point of view, homogenization procedure is a spectral threshold effect. General results are applied to periodic operators of mathematical physics (acoustics, elasticity, Maxwell, Schrödinger, Pauli).
Also, we will discuss recent results [4], [5] on homogenization of the operator A_ε in a bounded domain with the Dirichlet or Neumann boundary conditions. The operator error estimates for such operators can be proved by using the results in $\mathbb{R}^d$, introduction of the boundary layer correction term and a careful analysis of this term.

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Ivan Veselić, Martin Tautenhahn and Michela Egidi
Uncertainty Relations and Applications to the Schrödinger and Heat Conduction Equations

In four lectures we discuss unique continuation principles for various classes of functions, their relation to uncertainty principles, and their application in the analysis of certain elliptic and parabolic partial differential equations. We are in particular interested in domains and coefficient functions which have a multiscale structure as it is typical for periodic and random Schrödinger operators.
The first two lectures are held by Ivan Veselic, the third by Martin Tautenhahn, and the last by Michela Egidi.

Unique continuation properties and vanishing order (by Ivan Veselic)

We discuss classes of functions which satisfy the unique continuation principle (UCP) and which have finite vanishing order. This includes solutions to Laplace and Schroedinger equations on compact manifolds as studied by Donnelly & Fefferman, Kukavica, and Bakry. We introduce Carleman estimates as a tool to study the vanishing order.

Scale free unique continuation estimates (by Ivan Veselic)

Periodic and random Schroedinger operators have an underlying geometric lattice structure with a microscopic and a macroscopic length scale. For this reason one is interested in UCP which respect this structure. We dub them scale free UCP. We discuss several types of such scale free UCP, some aspects of the proof and some applications in the theory of random Schrödinger operators.

Quantitative unique continuation and application to control theory for the heat equation (by Martin Tautenhahn)

We discuss the application of quantitative scale free unique continuation principles to the control theory of the heat equation. This includes an explicit estimate of the control cost in terms of the characteristic scales of the problem. Moreover, we present extensions of the scale free unique continuation principle to elliptic operators with variable coefficients. This relies on a quantitative Carleman estimate for elliptic operators with variable coefficients, which we present as well.

Scale free Logvinenko-Sereda Theorem and applications (by Michela Egidi)

We present an uncertainty principle for functions on multidimensional tori analogous to Theorems of Logvinenko & Sereda and Kovrijkine. They allow us to give explicit estimates on the control cost for the heat equation. Further applications are discussed as well.

Talks

Ivgenii Afanasiev
On the Second Mixed Moment of Characteristic Polynomials of Sparse Hermitian Random Matrices

We consider asymptotics of the second mixed moment or correlation function of characteristic polynomials of sparse hermitian random matrices

$$M_n = (d_{jk}w_{jk})_{j,k=1}^{n}$$

where

$$d_{jk} = p^{-1/2}\begin{cases}1 \text{ with probability } \frac{p}{n};\\ 0 \text{ with probability } 1-\frac{p}{n}\end{cases}$$

and $2\Re w_{jk}$, $2\Im w_{jk}$, w_{ll} are i.i.d. Gaussian random variables with zero mean and unit variance.
It is shown that for finite p the second correlation function demonstrates a kind of transition: when $p < 2$ it factorizes in the limit $n \to \infty$, while for $p > 2$ there appears an interval $(-\lambda_*(p), \lambda_*(p))$ such that for $\lambda_0 \in (-\lambda_*(p), \lambda_*(p))$ the second correlation function behaves like that for Gaussian unitary ensemble (GUE), while for λ_0 outside the interval the second correlation function is still factorized. For $p \to \infty$ there is also a threshold in the behavior of the second correlation function near $\lambda_0 = \pm 2$: for $p \ll n^{2/3}$ the second correlation function factorizes, whereas for $p \gg n^{2/3}$ it behaves like that for GUE.

Kyrylo Andreiev
Regularized Integrals of Motion for the Korteweg-de Vries Equation with Steplike Initial Data

We construct an infinite series of the regularized integrals of motion for the Korteweg - de Vries equation $\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$, with steplike initial profile $u_0(x)$, which belongs to the Schwartz type class in the following meaning:

$$\int_{\mathbb{R}_+} x^m(|u_0(-x) - c^2| + |u_0(x)|)dx + \int_{\mathbb{R}} |x|^m |u_0^{(s)}(x)|dx < \infty,$$

for all integer $m \geq 1$ and $s \geq 1$. We propose a representation of these integrals of motion via the scattering data of the initial profile. Our work initiated by a generalization of the well known result of Faddeev and Zakharkov [1].

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Andrei Badanin
Inverse Problems and Sharp Eigenvalue Asymptotics for Euler-Bernoulli Operators

We consider Euler-Bernoulli operators with real coefficients on the unit interval. We prove the following results:

i) Ambarzumyan type theorem about the inverse problems for the Euler-Bernoulli operator.

ii) The sharp asymptotics of eigenvalues for the Euler-Bernoulli operator when its coefficients converge to the constant function.

iii) The sharp eigenvalue asymptotics both for the Euler-Bernoulli operator and fourth order operators (with complex coefficients) on the unit interval at high energy.

Denis Borisov
Boundary Homogenization: Norm Resolvent Convergence and Asymptotics

We discuss several two-dimensional models of quantum waveguides with several perturbations borrowed from the theory of boundary homogenization. The waveguides are modeled by planar strips, while the perturbations are frequent alternation of boundary conditions, fast oscillating boundary, perforation along a fixed curve. The operators are second order self-adjoint scalar elliptic operators. We classify possible homogenized operators and all the cases we prove the norm resolvent convergence of the considered operator to a homogenized one. We also establish the estimates for the rate of convergence. In some periodic cases we also construct asymptotic expansions for the spectrum of the perturbed operator.

Mark Dorodnyi
Homogenization of Hyperbolic-type Equations

In $L_2(\mathbb{R}^d;\mathbb{C}^n)$, we consider self-adjoint strongly elliptic second order differential operators $\mathcal{A}_\varepsilon$ with periodic coefficients depending on $\mathbf{x}/\varepsilon$. We study the behavior of the operator $\cos(\tau\mathcal{A}_\varepsilon^{1/2})$, $\tau \in \mathbb{R}$, for small ε. Approximations for this operator in the $(H^s \to L_2)$-operator norm are obtained. The method is based on the scaling transformation, the Floquet-Bloch theory, and the analytic perturbation theory. The results are applied to study the behavior of the solution $\mathbf{v}_\varepsilon$ of the Cauchy problem for the hyperbolic-type equation $\partial_\tau^2\mathbf{v}_\varepsilon = -\mathcal{A}_\varepsilon\mathbf{v}_\varepsilon + \mathbf{F}$. Applications to the acoustics equation and the system of elasticity theory are given.
The talk is based on the joint work with T.A.Suslina

Mariia Filipkovska
Global Solvability and Lagrange Stability of Semilinear Differential-algebraic Equations and Applications to Nonlinear Radio Engineering

The systems of differential-algebraic equations which in a vector form have the representation as the semilinear differential-algebraic equation (DAE) $d/dt[Ax(t)] + Bx(t) = f(t,x)$ are considered. The operators A, B to which $m \times n$ matrices A, B correspond may be degenerate. The theorems on the global solvability, the Lagrange stability and instability of the semilinear differential-algebraic equations (DAEs) with regular and singular characteristic pencils are presented. The theorems on the Lagrange stability give sufficient conditions of the existence and the uniqueness of the global solution bounded on the whole domain. The theorems on the Lagrange instability give sufficient conditions of the existence and the uniqueness of the solution with finite escape time (the solution is defined on a finite interval and unbounded). In the case when the characteristic pencil of the DAE is singular the corresponding system of differential-algebraic equations may be underdetermined or overdetermined. The theorems contains no constraints of the global Lipschitz condition type, which allows to use them to solve more general classes of applied problems. The mathematical models of nonlinear radio engineering devices are considered as applications.

Ivan Gurianov
Asymptotics of Resonant Tunneling in Quantum Waveguides with two Resonators

We consider the Helmholtz equation in a cylindric domain Ω with three narrows of small diameter ε. We set the Dirichlet boundary condition on $\partial\Omega$ and intrinsic radiation conditions at ∞. The parts of the domain between two neighbouring narrows play the role of resonators. In such a waveguide, the resonant tunneling can occur, i.e. the transmission coefficient $T(k)$ has peaks at some "resonant" wave numbers k. We present asymptotic formulas for wave functions as the diameters ε of the narrows tend to zero. We also obtain asymptotics of the transmission coefficient $T(k)$ and of the resonant wave numbers. The behaviour of $T(k)$ near a resonance is analysed. This is a joint work with O. Sarafanov.

Amru Hussein
Non-self-adjoint Graphs

On finite metric Graphs Laplace Operators subject to various classes of non-self-adjoint boundary conditions imposed at graph Vertices are considered. Spectral properties are investigated, in particular similarity transforms to self-adjoint operators. Concrete examples are discussed exhibiting that non-self-adjoint boundary conditions can yield to unexpected spectral features.

Andrii Khradbustovskyi
Spectral Properties of Domains with "Room-and-Passage" Boundary

Let Ω be a bounded domain in $\mathbb{R}^n$ ($n \geq 2$). We perturb it to a domain Ω^ε ($\varepsilon > 0$ is a small parameter) by attaching a family of small protuberances. Each protuberance consists of two parts – a small square ("rooms") and a narrow rectangle ("passage") connecting the "room" with Ω. The diameters of protuberances are of order $\mathcal{O}(\varepsilon)$; they are attached periodically, with a period ε, along a flat part of $\partial\Omega$ (and, thus, the number of attached protuberances tends to infinity as $\varepsilon \to 0$).
Peculiar spectral properties of so perturbed domains were observed for the first time by R. Courant and D. Hilbert [4, Chapter VI, § 2.6]. They considered the case of *one* attached protuberance and addressed the question how such a perturbation of the domain Ω influences the spectrum of the corresponding Neumann Laplacian. Later the aforementioned problem was treated in [1], where several (but *finitely many*)

protuberances are attached and more general geometry of "rooms" and "passages" is allowed.
In the current talk we study the spectral properties of the operator

$$\mathcal{A}^\varepsilon = -\varrho^\varepsilon \Delta_{\Omega^\varepsilon},$$

where $\Delta_{\Omega^\varepsilon}$ is the Neumann Laplacian in Ω^ε, the function $\varrho^\varepsilon(x)$ satisfies $C_1^\varepsilon \leq \varrho^\varepsilon(x) \leq C_2^\varepsilon$, $C_1^\varepsilon, C_2^\varepsilon$ are positive constants, moreover $\rho^\varepsilon = 1$ in Ω.
We prove that the spectrum of $\mathcal{A}^\varepsilon$ converges to the spectrum of some operator $\mathcal{A}$ acting in $L_2(\Omega) \times L_2(\Gamma)$, where Γ is a perturbed part of $\partial\Omega$. This operator is associated with the following spectral problem in Ω:

$$-\Delta u = \lambda u \text{ in } \Omega, \quad \frac{\partial u}{\partial n} + \mathcal{F}(\lambda) u = 0 \text{ on } \Gamma, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \setminus \Gamma,$$

where n is the outward-pointing unit normal, $\mathcal{F}(\lambda)$ is either rational (with one pole) or linear function. In the former case, the pole of $\mathcal{F}$ is a point of accumulation of eigenvalues of $\mathcal{A}$. The form of $\mathcal{F}$ depends on the geometry of protuberances and on the behaviour of ϱ^ε on the union of "rooms".
We also discuss the case, when Ω is an unbounded strip ("waveguide"), and prove that "rooms-and-passages" perturbation of Ω may cause the opening of spectral gaps. This is a joint work with G. Cardone (University of Sannio) [2], [3].

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Vitalii Marchenko
Infinitesimality of Operators with Non-basis Family of Eigenvectors

In a joint work with Prof. Dr. Grigory M. Sklyar we study the property of infinitesimality for linear operators possessing the following properties. Operators have simple purely imaginary eigenvalues, which essentially cluster at the infinity, and corresponding eigenvectors are dense, minimal, but not uniformly minimal, hence do not form a Schauder basis. For this purpose we introduce some special Hilbert and Banach spaces and apply the discrete Hardy inequality. We found conditions guaranteeing that the operator will be an infinitesimal generator of the C_0-group as well as conditions under which the operator will not generate even a C_0-semigroup. It turned out that this phenomenon essentially depends on the character of the asymptotic behavior of eigenvalues at the infinity. The results concerning the well/ill-posedness of the corresponding Cauchy problems were also obtained. The results obtained complement the remarkable results of G.Q. Xu & S.P. Yung (2005) and H. Zwart (2010) on the Riesz basis property for spectral families.

Yulia Meshkova
Homogenization of elliptic and parabolic Dirichlet problems in a bounded domain

The talk is based on a joint work with T. A. Suslina.
Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain of class $C^{1,1}$. In $L_2(\mathcal{O};\mathbb{C}^n)$, we consider a self-adjoint second order elliptic differential operator $B_{D,\varepsilon}$ with the Dirichlet boundary condition. The coefficients of $B_{D,\varepsilon}$ are periodic and depend on $\mathbf{x}/\varepsilon$; so, they oscillate rapidly as $\varepsilon \to 0$. We obtain approximations for the resolvent $(B_{D,\varepsilon} - \zeta I)^{-1}$ and for the semigroup $\exp(-B_{D,\varepsilon}t)$, $t \geq 0$, both in the $(L_2 \to L_2)$- and $(L_2 \to H^1)$-norms. The results of such type are called operator error estimates in homogenization theory.

Maria Platonova
Symmetric α-stable distributions for noninteger $\alpha > 2$ and associated stochastic processes

We construct analogues of symmetric α-stable distributions for noninteger indices $\alpha > 2$ and investigate their links to solutions of the Cauchy problem for some evolution equations.

Aleksandr Poretskii
The Maxwell system in waveguides with non-homogeneous filling

A waveguide occupies a domain $G \subset \mathbb{R}^3$ coinciding, outside a large ball, with the union of finitely many semicylinders $\Pi_+^r = \{(y^r, t^r) : y^r \in \Omega^r, t^r > 0\}$, $r = 1, \ldots, T$.

A dielectric permittivity ε and a magnetic permeability μ are matrix-valued functions in $\overline{G}$ that are smooth and positive definite. For $x = (y^r, t^r) \in \Pi^r_+$ and $t^r \to +\infty$, the matrices $\varepsilon(y^r, t^r)$ and $\mu(y^r, t^r)$ tend, at exponential rate, to $\varepsilon^r(y^r)$ and $\mu^r(y^r)$, respectively; the ε^r and μ^r can be arbitrary matrix-valued functions being smooth and positive definite on $\overline{\Omega^r}$.
In such a waveguide we consider the stationary Maxwell system with a real spectral parameter and conductive boundary conditions. We propose and justify a well-posed statement of the boundary value problem with intrinsic radiation conditions and describe asymptotics of solutions to the problem at infinity. Moreover, on the problem continuous spectrum we introduce a scattering matrix and prove that it is unitary.
To establish the results we extend the Maxwell system to an elliptic boundary value problem. Then we investigate the latter in detail and apply to it the results of the theory of elliptic boundary value problems in domains with several cylindrical ends. Finally, we derive the information about the original Maxwell system from the results, obtained for the elliptic problem. In particular, the solution of the elliptic problem with a right-hand-side, subject to some compatibility conditions, provides a solution to the original problem. The scattering matrix of the elliptic problem turns out to be block-diagonal and one of its blocks plays the role of the scattering matrix for the original Maxwell system.
The talk exposes some results of a joint research with B.A. Plamenevskii. The results generalize and develop those in [1], where empty electromagnetic waveguides were considered.

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Melchior Wirth
Does Diffusion Determine the Geometry?

Can one hear the shape of a drum? In mathematical terms this famous question of M. Kac asks whether two unitarily equivalent Laplacians live on the same geometric object. It is now known, that the answer to this question is negative in general. Following an idea of Wolfgang Arendt, we replace the unitary transformation intertwining the Laplacians by an order preserving one and then ask how much of the geometry is preserved. In this situation the associated semigroups, which encode diffusion, are equivalent up to an order isomorphism. Therefore, our question becomes as stated in the title and we try to give an answer. In particular, we discuss the situation for graph Laplacians.
This is joint work with Matthias Keller, Daniel Lenz and Marcel Schmidt.

Christoph Schumacher
The Anderson Model on the Bethe Lattice: Lifshitz Tails

The Anderson model is the prototypical random Schroedinger operator and used to model crystals with impurities and alloys. Mathematically, the Anderson Hamiltonian is the sum of the Laplace operator and a random potential. In this talk, we deal with the discrete Anderson operator on the Bethe lattice, which is the infinite tree graph with constant degree. Our interest is in the behaviour of the Integrated Density of States (IDS) of the Anderson Hamiltonian at the bottom of the spectrum. More specifically, we prove Lifshitz tail behavior, i.e. doubly exponential decay with exponent 1/2. Due to the exponential volume growth of the balls, the spectral theory of the free Laplace operator on the Bethe lattice fundamentally differs from Euclidian lattices. As a consequence, the elegant proof of Lifshitz tails in Euclidian lattices fails on the Bethe lattice. Instead, we study the Laplace transform of the IDS, employ Tauberian theorems, a discrete Feynman-Kac formula, a discrete IMS localization formula, spectral theory of finite symmetric rooted trees, an uncertainty principle for low-energy states, and epsilon-net argument and concentration inequalities. In the talk, I will present part of the proof. The full proof can be found here: `https://www.tu-chemnitz.de/mathematik/preprint/2014/PREPRINT.php?year=2014&num=17` This is joint work with Francisco Hoecker-Escuti.

Albrecht Seelmann
On the subspace perturbation problem

The variation of closed subspaces associated with isolated components of the spectra of linear self-adjoint operators under a bounded additive perturbation is considered. Of particular interest is the least restrictive condition on the norm of the perturbation that guarantees that the maximal angle between the corresponding subspaces is less than $\pi/2$. This problem has been discussed by different authors and is still unsolved in full generality. We give a survey on the most common approaches and recent developments.

Nikita Senik
On Homogenization for Periodic Elliptic Operators on an Infinite Cylinder

We consider an elliptic differential operator $\mathcal{A}^\varepsilon$ on $L_2(\mathbb{R}^{d_1} \times \mathbb{T}^{d_2})$ (where $d_1 > 0$ and $d_2 \geq 0$) of the form $\mathcal{A}^\varepsilon = -\operatorname{div} A(\varepsilon^{-1}x_1, x_2)\nabla$. Here A is periodic in the first variable and smooth in a sense in the second. We do not require $A(x)$ to be Hermitian, so the operator $\mathcal{A}^\varepsilon$ is not generally self-adjoint. The goal is to study the behavior of $(\mathcal{A}^\varepsilon - \mu)^{-1}$, with appropriate μ, as ε goes to 0. In this talk, we present approximations in the operator norm for $(\mathcal{A}^\varepsilon - \mu)^{-1}$ and $\nabla(\mathcal{A}^\varepsilon - \mu)^{-1}$ with error of order ε, as well as an approximation for $(\mathcal{A}^\varepsilon - \mu)^{-1}$ with error of order ε^2. The approximations involve the effective operator and two different correctors.

Ekaterina Shchetka
Complex WKB Method for Difference Equations in Unbounded Domains

In the complex plane, we consider the difference Shrödinger equation

$$\psi(z+h) + \psi(z-h) + v(z)\psi(z) = E\psi(z), \quad z \in \mathbb{C},$$

where $h > 0$ and $E \in \mathbb{C}$ are parameters, and v is a trigonometric polynomial, i.e., $v(z) = \sum_{k=-m}^{n} c_k e^{ikz}, m, n > 0, c_n, c_{-m} \neq 0$. We develop an asymptotic method to study solutions of this equation for small positive h.
The talk is based on a joint work with A. Fedotov.

Olena Sivak
Asymptotic Approximations for Solutions to Elliptic Boundary-value Problems in Perforated Domains

In this talk I will focus on asymptotic approximations for solutions to boundary-value problems for the second order elliptic differential operator with rapidly oscillating coefficients in domains which periodically perforated by small holes of the same order as the period of perforation. On the boundaries of the holes different types of boundary conditions, including nonlinear, are imposed.

Vladimir Sloushch
Asymptotics of discrete spectrum of periodic Schrödinger operator perturbed by non-negative potential and related estimates of singular values for integral operators

In $L_2(\mathbb{R}^d)$, $d \geq 1$, consider a periodic Schrödinger operator H with non-constant metric. Let V be the operator of multiplication by a nonnegative function $V(x)$ decaying at infinity. Let (α, β) be an *inner* gap in the spectrum of H, and let $\lambda \in [\alpha, \beta]$ be a fixed

point. The spectrum of the operator $H(t) := H + tV$, $t > 0$, inside the gap (α, β) is discrete. Denote by $N(\lambda, \tau)$ the number of eigenvalues of the operator $H(t)$ that have passed the point λ as t increased from 0 to τ. The asymptotics of $N(\lambda, \tau)$ for $\tau \to +\infty$ is obtained in the case where the perturbation $V(x)$ has power-like asymptotics at infinity: $V(x) \sim \omega(x/|x|)|x|^{-\rho}$, $|x| \to +\infty$, $\rho > 0$. The main result can be represented in the following form: $N(\lambda, \tau) \sim \Gamma_\rho(\lambda)\tau^{d/\rho}$, $\tau \to +\infty$. Here the coefficient $\Gamma_\rho(\lambda)$ is computed in terms of *the band functions* of the operator H. Under certain conditions, this asymptotics holds at the left edge of the gap $\lambda = \alpha$ as well. We impose no additional restrictions on the smoothness of the coefficients of the operator H. The derived asymptotics is non-local with respect to energy, its order is different from the "standard" $\tau^{d/2}$. The Weyl nature of the asymptotics reveals itself if the roles of coordinates and quasi-momenta are switched.
The verification of the main result is based on analysis of the asymptotics of singular numbers of certain integral operators. Down this route, we employ different generalizations of the Cwikel estimate. In particular, we build Cwikel type estimate for the singular values of the operator $f(H)g(x)$ if $f(H)g(x)$ is Hilbert-Schmidt operator; in addition, estimate for the singular values of the operator $\varphi(H)V(x) - V(x)\varphi(H)$ is established for appropriate φ.

Kateryna Stiepanova
Extinction of Solutions for quasi-linear Parabolic Equations

Investigations are devoted to the study of the extinction of solutions in finite time to initial-boundary value problems for a wide classes of nonlinear parabolic equations of the second and higher orders with a degenerate absorption potential, whose presence plays a significant role for the mentioned nonlinear phenomena.
So, behavior of solutions to the parabolic equation of non-stationary diffusion with double nonlinearity and a degenerate absorption term:

$$\left(|u|^{q-1}u\right)_t - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(|\nabla_x u|^{q-1}\frac{\partial u}{\partial x_i}\right) + a_0(x)|u|^{\lambda-1}u = 0 \quad \text{in} \quad \Omega \times (0, T),$$

where Ω is bounded domain in $\mathbb{R}^N$, $N \geqslant 1$, $0 \in \Omega$, $a_0(x) \geq d_0 \exp\left(-\frac{\omega(|x|)}{|x|^{q+1}}\right)$, $x \in \Omega \setminus \{0\}$, $d_0 = const > 0$, $0 \leq \lambda < q$, $\omega(\cdot) \in C([0, +\infty))$, $\omega(0) = 0$, $\omega(\tau) > 0$ when $\tau > 0$ was studied. As well known the extinction property means that any solution of the mentioned equation vanishes in Ω in a finite time. Modifying the local energy approach of [1], we obtain a condition of Dini type on the function $\omega(\cdot)$ that ensures the extinction.
Also we investigate the property of extinction in the finite time of solutions to the initial-boundary problem for $2m$ order nonlinear parabolic equation with absorption

of the following type:

$$\left(|u|^{q-1}u\right)_t + (-1)^m \sum_{|\eta|=m} D_x^\eta \left(|D_x^m u|^{q-1} D_x^\eta u \right) + a(x)|u|^{\lambda-1}u = 0 \ \text{ in } \ \Omega \times (0,+\infty),$$

where Ω is bounded domain in $\mathbb{R}^N$, $N \geqslant 1$, $0 \in \Omega$, $m \geq 1$, $0 \leq \lambda < q$, an absorption potential $a(x)$ is nonnegative, measurable, bounded in Ω function. Using the semi-classical technic of [2], we find sufficient conditions, which guarantee the extinction for the mentioned equation above. These conditions are depending on N, m, and on the parameter of homogeneous nonlinearity of the main part in the equation q.

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Daria Tieplova
Distribution of Eigenvalues of Sample Covariance Matrices with Tensor Product Samples

We consider real symmetric and hermitian random matrices

$$M_n = \frac{1}{n^2} \sum_{\mu=1}^{m} X^\mu \otimes \bar{X}^\mu,$$

where $X^\mu = B(Y^\mu \otimes Y^\mu)$ and $\{Y^\mu\}_{\mu=1}^m$ are i.i.d. random vectors of $\mathbb{C}^n$ such that:

$$\mathbb{E}\{Y_i^\mu\} = \mathbb{E}\{Y_i^\mu Y_k^\nu\} = 0, \, \mathbb{E}\{Y_i^\mu \bar{Y}_k^\mu\} = \delta_{ik},$$

and $B = \{B_{\bar{i},\bar{j}}\}$ is an $n^2 \times n^2$ non-random hermitian matrices, where $\bar{i} = (i_1, i_2)$ and $\bar{j} = (j_1, j_2)$ are multi-indexes. Introduce the $n^2 \times n^2$ matrix

$$J_{\bar{p},\bar{q}} = \delta_{\bar{p}\,\bar{q}} + \delta_{\bar{p}'\bar{q}},$$

and denote by N_n and σ_n the Normalized Counting Measure of eigenvalues of M_n and BJB respectively. The main result is

Theorem 1 *Assume that the sequence σ_n converges weakly to a probability measure σ:*

$$\lim_{n\to\infty} \sigma_n = \sigma,$$

B is bounded uniformly in n, and $\{m_n\}$ is a sequence of positive integers such that

$$m_n \to +\infty,\ n \to +\infty,\ c_n = m_n/n^2 \to c \in [0, +\infty).$$

Then the Normalized Counting Measures N_n of eigenvalues of M_n converge weakly in probability to a non-random probability measure N, and if $f^{(0)}$ is the Stieltjes transform of σ, then the Stieltjes transform f of N is uniquely determined by the equation

$$f(z) = f^{(0)}\left(\frac{z}{c - zf(z) - 1}\right)(c - zf(z) - 1)^{-1}.$$

in the class of Stieltjes transforms of probability measures.

Christoph Uebersohn
On the difference of spectral projections

For a bounded self-adjoint operator T and a self-adjoint operator S of rank one acting on a complex separable Hilbert space of infinite dimension, we study the difference

$$D(\lambda) := E_{(-\infty,\lambda)}(T+S) - E_{(-\infty,\lambda)}(T), \quad \lambda \in \mathbb{R},$$

of the spectral projections with respect to the open interval $(-\infty, \lambda)$. We show that in most cases, the difference $D(\lambda)$ is unitarily equivalent to a bounded self-adjoint Hankel operator. This is motivated by a classical example given by Mark Kreĭn, where the difference $D(\lambda)$ is equal to a Hankel integral operator on $L^2(0,\infty)$ for all λ in $\mathbb{R}$.