Uncertainty Relations and Applications for the Schrödinger and Heat conduction equation

Ivan Veselić

TU Chemnitz $\xrightarrow{\text{on the move}}$ TU Dortmund

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based on joint works/projects with

Denis Borisov, Michela Egidi, Ivica Nakić, Constanza Rojas-Molina, Matthias Täufer, Martin Tautenhahn

Uncertainty Principles in Fourier Analysis

Let $f : \mathbb{R} \to \mathbb{C}$ have well-defined Fourier transform $\hat{f} : \mathbb{R} \to \mathbb{C}$

$$\hat{f} : \mathbb{R} \to \mathbb{C}, \quad \hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) \mathrm{e}^{-2\pi i x \xi} \mathrm{d}x$$

Decay properties of $f \implies$ smoothness properties of $\hat{f}$:

If f is	then $\hat{f}$ is
Square integrable	square integrable (*Plancherel's Theorem*).
Absolutely integrable	continuous (*Riemann-Lebesgue lemma*).
Rapidly decreasing	smooth (*Theory of Schwartz functions*).
Exponentially decaying	analytic in a strip.
Compactly supported	entire and at most exponential growth (*Paley-Wiener theorem*).

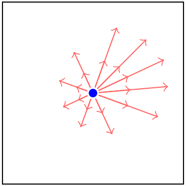
The last two relations can be seen as manifestations of the *Uncertainty Principle*:

f strongly localized in space $\Rightarrow$ $\hat{f}$ widely dispersed in space,

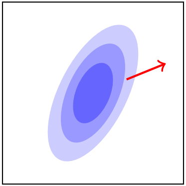
i.e. f and $\hat{f}$ cannot both decay too strongly at infinity unless $f = 0$.

Quantum Theory: Impossible to measure position and momentum simultaneously with arbitrary precision

Heisenberg Uncertainty Principle (1927):
Lower bound on volume occupied by particle in phase space $\sigma_x \sigma_p \geq 2\hbar$



σ_x small, but σ_p large



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Mathematically speaking: If $\int |f(x)|^2 \mathrm{d}x = \int |\hat{f}|^2 \mathrm{d}\xi = 1$, then

$$\left(\int |x|^2 |f(x)|^2 \mathrm{d}x \right) \cdot \left(\int |\xi|^2 |\hat{f}(\xi)|^2 \mathrm{d}\xi \right) \geq \frac{1}{(4\pi)^2}.$$

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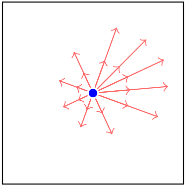

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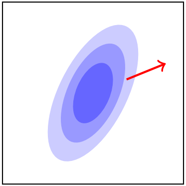

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If $f(x) = \mathrm{e}^{-\pi a x^2}$ Gaussian $\Longrightarrow \hat{f}(\xi) = \frac{1}{\sqrt{a}} \mathrm{e}^{-\pi \xi^2 / a}$. Both decay faster than exponentially.

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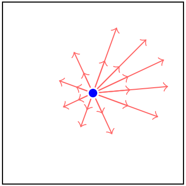

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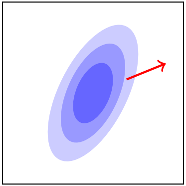

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If $f(x) = \mathrm{e}^{-\pi a x^2}$ Gaussian $\Longrightarrow$ $\hat{f}(\xi) = \frac{1}{\sqrt{a}} \mathrm{e}^{-\pi \xi^2 / a}$. Both decay faster than exponentially. This is fastest possible simultaneous decay for f and $\hat{f}$!

Theorem (Hardy's Uncertainty Principle)

Let $|f(x)| \leq C\mathrm{e}^{-\pi a x^2}$ and $|\hat{f}(\xi)| \leq C\mathrm{e}^{-\pi \xi^2 / a}$. Then $f(x)$ is a scalar multiple $e^{-\pi a x^2}$.

Support condition: annihilating pairs [Nazarov'93, Jaming'07]

Let $S, \Sigma \subset \mathbb{R}^d$ of finite Lebesgue measure and $f \in L^2(\mathbb{R}^d)$. Then

$$\|f\|_2^2 \leq Ce^{C|S|\cdot|\Sigma|} \left(\int_{S^c} |f^2| + \int_{\Sigma^c} |\hat{f}|^2 \right)$$

Thus S and Σ form an *strongly annihilating* pair.

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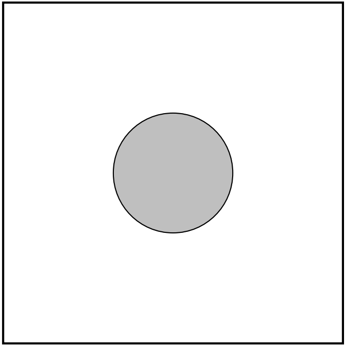
We have seen different manifestations of Uncertainty Principle

- Lower bound on variance of (quantum mechanical particle) density.
- Lower bound tail decay of density.
- Lower bound on measure of supports of f and $\hat{f}$: At least one of infinite measure.

Move now from holomophic functions to solutions of PDE

$B(0, \delta) \subset \Lambda_1 = (-1/2, 1/2)^d$
for some $\delta \in (0, 1/2)$



Consider
Schrödinger operator on cube Λ_1 with bounded, measurable potential $V \colon \Lambda_1 \to \mathbb{R}$.

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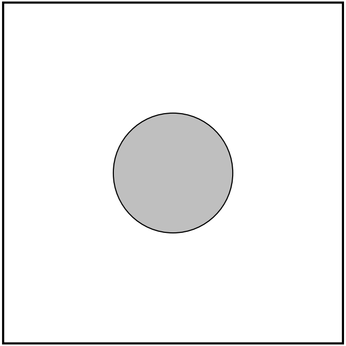

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Quantitative Unique continuation estimate

There $\exists\, M = M(d) \in (0,\infty)$, s. t. for all

- $\delta, K > 0$.
- $V : \Lambda_1 \to [-K, K]$
- $H = (-\Delta + V)_{\Lambda_1}$
- Dirichlet boundary conditions at $\partial\Lambda_1$
- $\psi \in W_0^{2,2}(\Lambda_1; \mathbb{R})$, $H\psi = 0$

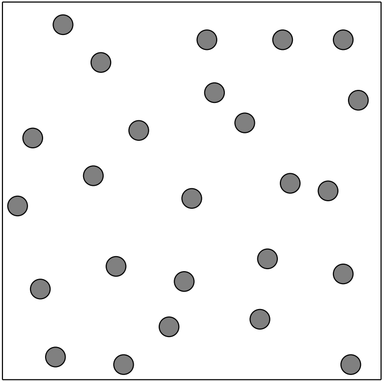
$$\Longrightarrow \int_{B(0,\delta)} |\psi|^2 \geq \delta^{M(1+K^{2/3})} \int_{\Lambda_L} |\psi|^2$$

Normalize $\|\psi\|_{L^2(\Lambda_1)} = 1$:
Control of vanishing order in L^2-sense/ quantitative unique continuation principle.

Actually, this is a simple case of a more powerful theorem, which we will present next.

Retrieval of global properties from local data

Let $\Lambda \subset \mathbb{R}^d$ be a region in space, $S \subset \Lambda$ a subset, and $f\colon \Lambda \to \mathbb{R}$.



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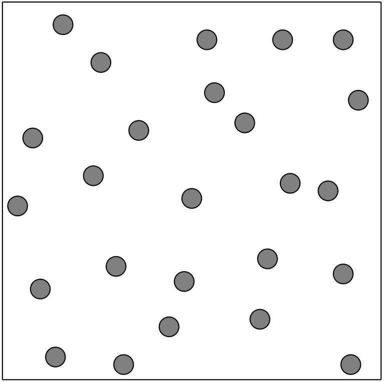

What can one say about certain properties of $f\colon \Lambda \to \mathbb{R}$ given certain properties of $f|_S\colon S \to \mathbb{R}$?

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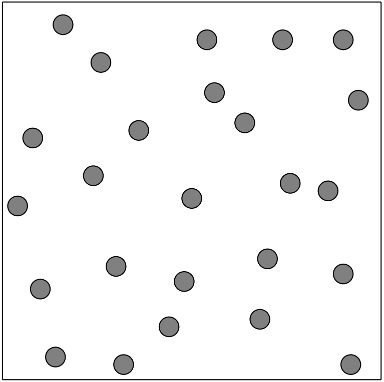

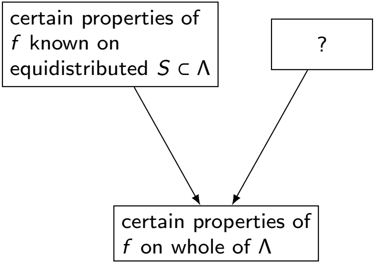

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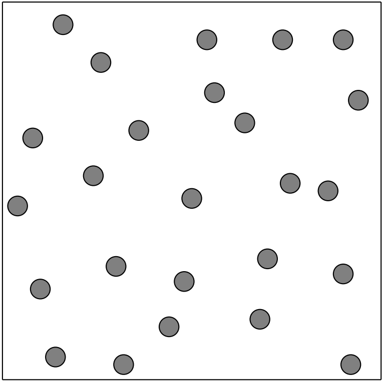

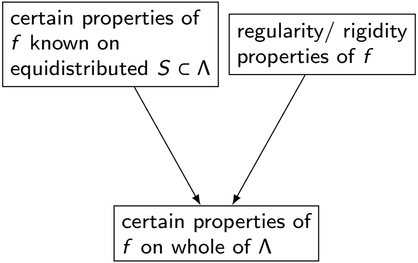

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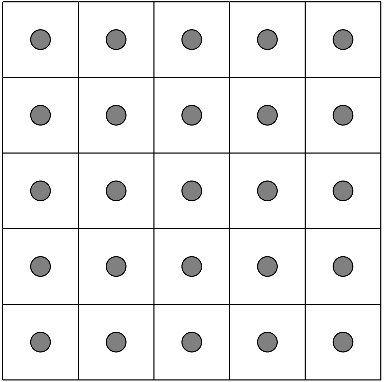
Geometrical aspect: Conditions to impose on subset S?

Notion of equidistributed subsets.



- a natural choice would be a periodic arrangement of balls
- equidistributed set could be seen as a generalization thereof
- small perturbations of periodic arrangement should be included as well
- S relatively dense in $\mathbb{R}^d$
  e. g. Delone set
- S subset with positive density

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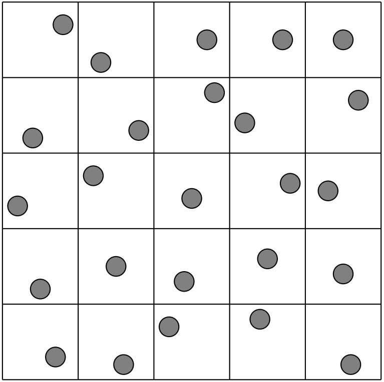

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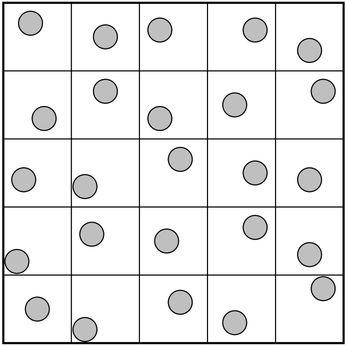
Scale-free unique continuation: motivated by spectral theory of random Schrödinger operators



- $\Lambda_L = (-L/2, L/2)^d$
- $S_L(\delta) = \Lambda_L \cap \left(\bigcup_{j \in \mathbb{Z}^d} B(x_j, \delta) \right)$
- $H_L = (-\Delta + V)_{\Lambda_L}$ with Dirichlet or periodic b.c. at $\partial\Lambda_L$

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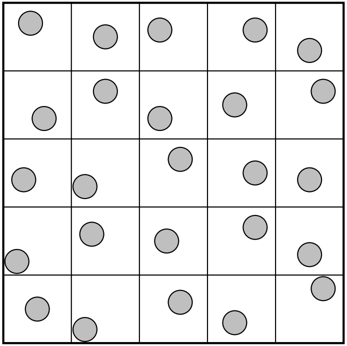

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Theorem [Rojas-Molina & Ves. 13]

There $\exists\, M = M(d) \in (0, \infty)$, s. t. for all

- $\delta, K > 0, \quad L \in \mathbb{N}$
- $H = (-\Delta + V)_{\Lambda_1}, \ V \colon \Lambda_1 \to [-K, K]$
- Dirichlet or periodic boundary conditions at $\partial \Lambda_1$
- $\psi \in W^{2,2}_{0/\mathrm{per}}(\Lambda_1; \mathbb{R}), \ H\psi = 0$

$$\Longrightarrow \int_{S_L(\delta)} \psi^2 \geq C_{UC} \int_{\Lambda_L} \psi^2, \quad C_{UC} := \delta^{M(1+K^{2/3})}$$

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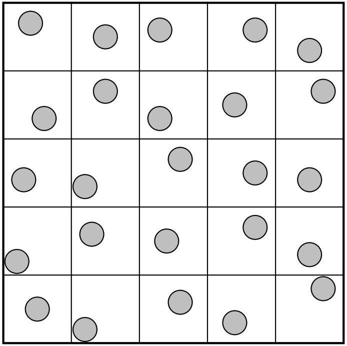

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quantitative dependence of C_{UC} on parameters

- independent of position of $B(x_j, \delta)$ within $\Lambda_1 + j$
- independent of scale $L \in \mathbb{N}$
- depends on V only through $\|V\|_\infty$ (on exponential scale)
- depends on $\delta > 0$ polynomially, $C_{UC} \geq \delta^N$, some $N \in \mathbb{N}$

Theorem holds for all eigenfunctions ψ

$$H_L\psi = E\psi \Leftrightarrow (H_L - E)\psi = 0$$

with possibly larger constant $K = K_{V-E}$ instead of $K = K_V$.

Question [Rojas-Molina & Ves. 13]

True for linear combinations $\psi \in \operatorname{Ran} \chi_{(-\infty,E]}(H_L)$ of eigenfunctions as well?

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More precisely:

Given $\delta > 0, K \geq 0, E \in \mathbb{R}$ is there a constant $C_{UC} > 0$ s. t. for all measurable $V\colon \mathbb{R}^d \to [-K, K]$, all $L \in \mathbb{N}$, and all sequences $(x_j)_{j\in\mathbb{Z}^d} \subset \mathbb{R}^d$ with $B(x_j,\delta) \subset \Lambda_1 + j$

$$\chi_{(-\infty,E]}(H_L)\, W_L\, \chi_{(-\infty,E]}(H_L) \geq C_{UC}\ \chi_{(-\infty,E]}(H_L), \quad \text{where } W_L = \chi_{S_L(\delta)}$$

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Slightly more general:

Given $\delta > 0, K \geq 0, a < b \in \mathbb{R}$ is there is a constant $C_{UC} > 0$ s. t. for all measurable $V\colon \mathbb{R}^d \to [-K, K]$, all $L \in \mathbb{N}$, and all sequences $(x_j)_{j\in\mathbb{Z}^d} \subset \mathbb{R}^d$ with $B(x_j, \delta) \subset \Lambda_1 + j$

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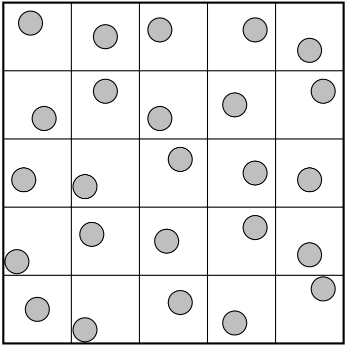
Theorem [Klein 13]

True for sufficiently small $\gamma = b - a$, depending on K, δ.

Sufficient for many questions in spectral theory of random Schrödinger operators.

Theorem [Nakić, Täufer, Tautenhahn & Ves. 15, 16]



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There $\exists M_0 = M_0(d)$ such that for all

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- $\alpha_k \in \mathbb{C}$, $\psi = \sum_{E_k \leq E} \alpha_k \psi_k$
- $(x_j)_{j \in \mathbb{Z}^d} \subset \mathbb{R}^d$, $B(x_j, \delta) \subset \Lambda_1 + j$

holds
$$\int_{S_L(\delta)} |\psi|^2 \geq C_{UC} \int_{\Lambda_L} |\psi|^2$$

with
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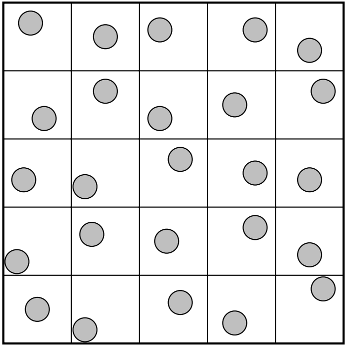
Theorem [Nakić, Täufer, Tautenhahn & Ves. 15, 16]



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Restriction to $E_k \in [E-1, E]$ does not improve estimate: Upper bound decisive.

Application to lifting of eigenvalues

Lifting lemma [NTTV]

$$\lambda_i(H_L + U_L) \geq \lambda_i(H_L) + \alpha C_{\mathrm{sfuc}}(d, \delta, E, \|V_L + U_L\|_\infty).$$

for
$\lambda_i(-H_L^t + B_L) \leq E$.

Application to lifting of eigenvalues

Lifting lemma [NTTV]

Let $\delta, Z, L, S_L(\delta), V, H_L, E$ as above. Let $\alpha > 0$ and $U_L \geq \alpha W_L = \alpha \chi_{S_L(\delta)}$ Then

$$\lambda_i(H_L + U_L) \geq \lambda_i(H_L) + \alpha C_{\mathrm{sfuc}}(d, \delta, E, \|V_L + U_L\|_\infty).$$

for all eigenvalues (counted in increasing order, with multiplicities) satisfying $\lambda_i(-H_L^t + B_L) \leq E$.

Scaled version of NTTV Result

Scaling

$B(x_j, \delta) \subset \Lambda_G + j = [-G/2, G/2]^d + j$

Schrödinger operator $H_L^t = -t\Delta + V$

Theorem [NTTV]

With K as above

$$\|\psi\|^2_{L^2(S_L(\delta))} \geq C^{G,t}_{\mathrm{sfuc}} \|\psi\|^2_{L^2(\Lambda_L)}$$

for any $\psi = \sum_{E_k \leq E} \alpha_k \psi_k \in \operatorname{ran} \chi_{(-\infty, E]}(H_L^t)$ where

$$C^{G,t}_{\mathrm{sfuc}} = C^{G,t}_{\mathrm{sfuc}}(d, \delta, E, \|V\|_\infty) := \left(\frac{\delta}{G}\right)^{K\left(1 + G^{4/3}\|V\|_\infty^{2/3}/t^{2/3} + G\sqrt{E/t}\right)}.$$

Scaled version of NTTV Result

Scaling

Let $G, t > 0$, $\delta \in (0, G/2)$, $Z = (x_j)_{j\in(G\mathbb{Z})^d} \subset \mathbb{R}^d$, s.t

$B(x_j, \delta) \subset \Lambda_G + j = [-G/2, G/2]^d + j$ Call sequence Z: (G, δ)-equidistributed

$S(\delta) = B(Z, \delta) = \bigcup_{j\in\mathbb{Z}^d} B(x_j, \delta)$

Let $L \in G\mathbb{N}$, $S_L(\delta) = \Lambda_L \cap S(\delta)$,
$V \in L^\infty(\mathbb{R}^d, \mathbb{R})$
Schrödinger operator $H_L^t = -t\Delta + V$ on Λ_L, $L \in \mathbb{N}$, with Dirichlet, Neumann, or periodic b.c.

Theorem [NTTV]

With K as above

$$\|\psi\|^2_{L^2(S_L(\delta))} \geq C^{G,t}_{\mathrm{sfuc}} \|\psi\|^2_{L^2(\Lambda_L)}$$

for any $\psi = \sum_{E_k \leq E} \alpha_k \psi_k \in \operatorname{ran} \chi_{(-\infty,E]}(H_L^t)$ where

$$C^{G,t}_{\mathrm{sfuc}} = C^{G,t}_{\mathrm{sfuc}}(d, \delta, E, \|V\|_\infty) := \left(\frac{\delta}{G}\right)^{K\left(1+G^{4/3}\|V\|_\infty^{2/3}/t^{2/3}+G\sqrt{E/t}\right)}.$$

Application to lifting of eigenvalues

Recall

Let $G, t > 0$, $\delta \in (0, 1/2)$, $Z = (x_j)_{j \in (G\mathbb{Z})^d} \subset \mathbb{R}^d$, s.t

$B(x_j, \delta) \subset \Lambda_G + j = [-G/2, G/2]^d + j$

$S(\delta) = B(Z, \delta) = \bigcup_{j \in \mathbb{Z}^d} B(x_j, \delta)$

Let $L \in G\mathbb{N}$, $S_L(\delta) = \Lambda_L \cap S(\delta)$, and $W_L = \chi_{S_L(\delta)}$,
$V \in L^\infty(\mathbb{R}^d, \mathbb{R})$
Schrödinger operator $H_L^t = -t\Delta + V$ on Λ_L, $L \in \mathbb{N}$, with Dirichlet, Neumann, or periodic b.c.

Lifting lemma [NTTV]

Let $G, t, \delta, Z, L, S_L(\delta), V, H_L^t, E$ as above. Let $\alpha > 0$ and $U_L \geq \alpha W_L = \alpha \chi_{S_L(\delta)}$ Then

$$\lambda_i(H_L^t + U_L) \geq \lambda_i(H_L^t) + \alpha C_{\mathrm{sfuc}}^{G,t}(d, \delta, E, \|V_L + U_L\|_\infty).$$

for all eigenvalues (counted in increasing order, with multiplicities) satisfying $\lambda_i(-H_L^t + B_L) \leq E$.

Equidistribution property of eigenfunctions

Context in the spectral theory of random Schrödinger operators

Physical intuition of Anderson phase transition

localized energy regime	**delocalized** energy regime
no propagation of wave packets	propagation of wave packets
Poissonian behaviour of rescaled eigenvalues	level repulsion
exponential/fast decay of eigenfunctions	**spread out eigensolutions**

Our theorem about equidistribution of (linear combinations of) eigenfunctions is **not** related to delocalization of eigensolutions, but an universal a-priori bound.

Interestingly, it is often used to when proving localization of eigenfunctions, which might seem paradoxical at first sight.

The relevance of the scale free unique continuation principle is illustrated by the fact that it has been studied before in numerous papers for particular situations.

Previously results for particular settings

vanishing potential $V \equiv 0$ low energies $0 < E \ll 1$ individual eigenfunctions periodic set Z	Kirsch 96: Brownian m. hitting probabilities Bourgain, Kenig 05: spatial averaging
as above, but non-periodic Z similar, implicit results	Germinet 08: spatial averaging Boutet de Monvel, Naboko, Stollmann, Stolz
periodic V energies at spectral edges individual eigenfunctions periodic set Z	Kirsch, Stollmann, Stolz 98 Combes, Hislop, Nakamura 01
eigenvalue lifting implies uncertainty relation at low energies	Boutet de Monvel, Lenz, Stollmann 11
$d = 1$, periodic V and Z for individual eigenfunctions extension to non-periodic V, Z	Veselic 96, Kirsch, Veselic 02 Helm, Veselic 07
for spectral projectors for periodic V and Z quantitative version	Combes, Hislop, Klopp 03, 07 Floquet-theory & compactness Germinet, Klein 13
our setting but for individual eigenfunctions extension for spectral projection of short energy intervals	Rojas-Molina, Veselic 13 Klein 13

Extension to magnetic Schrödinger operators: Efficient bounds only for **bounded** vector potentials.

Various names for such results

- scale free unique continuation principle:
  quantitative version of unique continuation principle, uniform on all large scales
- uncertainty relation:
  condition $\psi \in \operatorname{ran} \chi_{(-\infty,E]}(H_L)$ in momentum/Fourier-space enforces delocalization/flatness in direct space
- gain of positive definiteness:
  selfadjoint operator $W \geq 0$ has kernel
  for spectral projector P of Hamiltonian:
  restriction $P\,W\,P \geq c\,P$ is strictly positive
- in control theory: observability estimate or spectral inequality

Indirect relation to

- quantum ergodicity:
  equidistribution property of (properly chosen combinations of) eigenfunctions:
  comparison of measure $|\psi(x)|^2 \mathrm{d}x$ with uniform distribution
  Our result: worst case scenario for arbitrarily chosen linear combinations of eigenfunctions
- uniform uncertainty principle or restricted isometry property in compressed sensing and sparse recovery

Application to control theory of our scale free observability estimate

Consider heat equation with control function v:

$$\begin{cases} \partial_t y - \Delta y = \mathbf{1}_S v & \text{in } \Omega \times (0, T) \\ y = 0 & \text{on } \partial\Omega \times (0, T) \\ y(0) = y_0 & \text{in } \Omega \end{cases}$$

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Null-controlability: for each $T > 0$ there exists a $C(T)$ such that for all $y_0 \in L^2(\Omega)$ exists control $v \in L^2((0, T) \times \Omega)$ with

$$y(T) = 0 \quad \text{and} \quad \|v\|_{L^2((0,T)\times\Omega)} \leq C(T)\|y_0\|_{L^2(\Omega)}.$$

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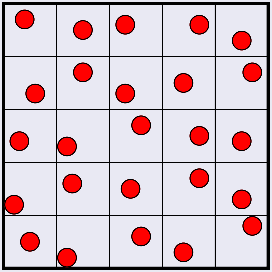
Our setting:



$$\Omega = \Lambda_L, \; S = W_\delta(L)$$

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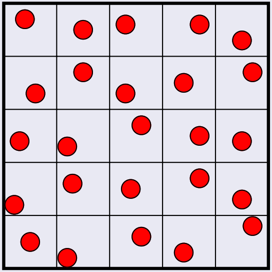


$\Omega = \Lambda_L$, $S = W_\delta(L)$

$\Rightarrow$ scale-free null-controlability ($C(T)$ is L-independent)

$\Rightarrow$ explicit δ-dependence

Variable coefficient 2nd order partial differential operator

Class considered

Let $d \in \mathbb{N}$ and $\mathcal{L}$ be the differential expression for $x \in \mathbb{R}^d$.

$$\mathcal{L}u := -\sum_{i,j=1}^{d} \partial_i \left(a^{ij} \partial_j u \right)$$

satisfying

- symmetry condition $\quad a^{ij}(x) = a^{ji}(x)$ for all $i, j \in \{1, \ldots, d\}$
- ellipticity condition $\quad \theta^{-1}|\xi|^2 \leq \sum_{i,j=1}^{d} a^{ij}(x)\xi_i\xi_j \leq \theta|\xi|^2$
- Lipschitz condition $\quad \sum_{i,j=1}^{d} |a^{ij}(x) - a^{ij}(y)| \leq t|x-y|$

Partial result for variable coefficient divergence type operators

Theorem [Borisov, Tautenhahn & Ves. 15]

$$\|\psi\|_{S_\delta}^2 \geq c_{\mathrm{sfuc}} \|\psi\|_{\mathbb{R}^d}^2 ,$$

where

$$c_{\mathrm{sfuc}} = d_1 \left(\frac{\delta}{d_2} \right)^{d_3 \left(1 + \|V\|_\infty^{2/3} + \|b\|_\infty^2 + \|c\|_\infty^{2/3} \right) - d_4}$$

Partial result for variable coefficient divergence type operators

Theorem [Borisov, Tautenhahn & Ves. 15]

Assume that

$$d^2 \cdot \theta_1^6 \cdot \theta_2 < 1/(99 \cdot \mathrm{e}).$$

Then for all measurable and bounded $V : \mathbb{R}^d \to \mathbb{R}$, all $\psi \in W^{2,2}(\mathbb{R}^d)$ satisfying $|\mathcal{L}\psi| \leq |V\psi|$ almost everywhere on $\mathbb{R}^d$, all $\delta \in (0, 1/2)$ and all $(1, \delta)$-equidistributed sequences we have

$$\|\psi\|_{S_\delta}^2 \geq c_{\mathrm{sfuc}} \|\psi\|_{\mathbb{R}^d}^2,$$

where

$$c_{\mathrm{sfuc}} = d_1 \left(\frac{\delta}{d_2} \right)^{d_3 \left(1 + \|V\|_\infty^{2/3} + \|b\|_\infty^2 + \|c\|_\infty^{2/3} \right) - d_4}$$

with $d_1, \dots, d_4$ depending only on d, θ_1, θ_2.

Carleman estimates

- allow to deduce ucp (after some calculations)
- whole zoo of Carleman estimates exists
- many with abstract weight functions
  (satisfying Hörmanders subellipticity condition)
- we want explicit estimate, thus explicit weight function

We start with a formulation of [Bourgain, Kenig 05] since this has given crucial stimulus to the theory of random Schrödinger operators.

Carleman estimate as formulated in [Bourgain, Kenig 05]

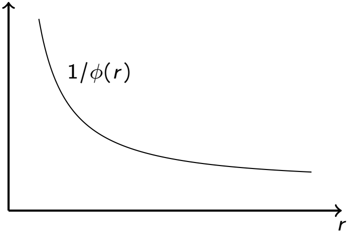
Weight function

$\phi\colon [0,\infty) \to [0,\infty)$

$\phi(r) =$

$r \exp\left(- \int_0^r \frac{1-\mathrm{e}^{-t}}{t} \mathrm{d}t \right)$

$w\colon \mathbb{R}^d \to [0,\infty)$, $w(x) = \phi(|x|)$



$\Rightarrow \forall\ r \in (0,1) : r/3 \le \phi(r) \le r$

Theorem [Bourgain & Kenig 05]

There are constants $C_1(d)$ and $C_2(d) \in [1,\infty)$ s. t. for all $\alpha \ge C_1$ and real valued $f \in C^2(B(0,1))$ with compact support in $B(0,1) \setminus \{0\}$ we have

$$\alpha^3 \int w^{-1-2\alpha} f^2 \,\mathrm{d}x \le C_2 \int w^{2-2\alpha} (\Delta f)^2 \,\mathrm{d}x$$

Concise, clear proof.

Following ideas of [Escauriaza & Vessella 03] for parabolic case with variable coefficients. Short proofs. Somewhat clearer with the help of Morassi, Rosset, & Vessella 11

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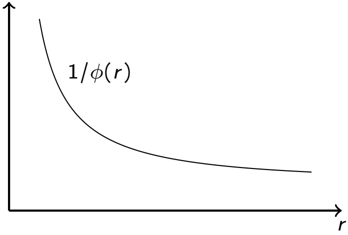

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Possible to scale the inequality to a ball of radius ρ and extend to Sobolev space H^2.

Consequences/Applications of Carleman estimates

Theorem [Bourgain & Klein 13]

Let $\Lambda \subset \mathbb{R}^d$ open, $V\colon \Lambda \to \mathbb{R}$ measurable, bounded, $\psi \in W^{2,2}(\Lambda, \mathbb{R})$ satisfying $-\Delta\psi + V\psi = 0$ a.s. For $\Theta \subset \Lambda$ bounded, measurable set

$$Q(x, \Theta) := \sup_{y \in \Theta} |y - x| \quad \text{for } x \in \Lambda \setminus \overline{\Theta}$$

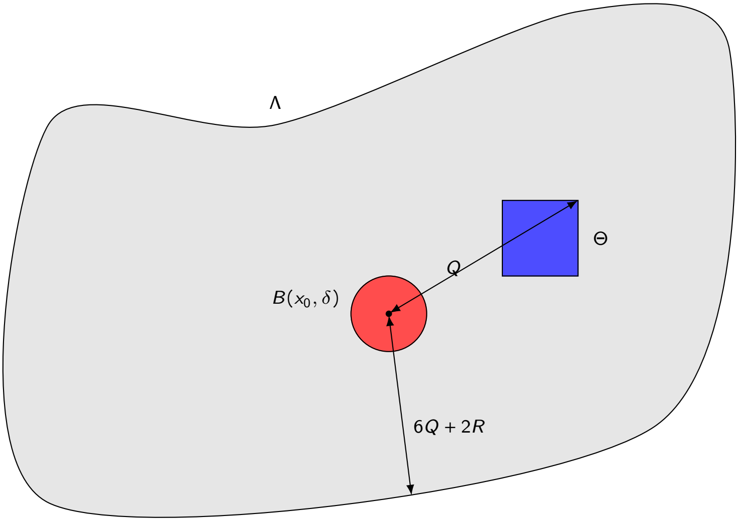
If $Q = Q(x_0, \Theta) \geq 1$, $\operatorname{dist}(x_0, \Theta) > 0$, $B(x_0, 6Q + 2) \subset \Lambda$ and $0 < \delta \leq \min\{\operatorname{dist}(x_0, \Theta), 1/24\}$, then

$$\|\psi\|^2_{L^2(\Theta)} \leq \left(\frac{Q}{\delta}\right)^{K\left(1 + \|V\|_\infty^{2/3}\right)\left(Q^{4/3} + \log \frac{\|\psi\|_{L^2(\Lambda)}}{\|\psi\|_{L^2(\Theta)}}\right)} \|\psi\|^2_{L^2(B(x_0, \delta))}$$

where $K > 0$ depends only on d.

Analogous results in [Germinet & Klein 13], [Rojas-Molina & Veselić 13]

Geometric assumptions



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$$\frac{\|\psi\|^2_{L^2(\Theta)}}{\|\psi\|^2_{L^2(B(x_0,\delta))}} \leq \left(\frac{Q}{\delta}\right)^{K\left(1+\|V\|_\infty^{2/3}\right)\left(Q^{4/3}+\log \frac{\|\psi\|_{L^2(\Lambda)}}{\|\psi\|_{L^2(\Theta)}}\right)}$$

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where $K > 0$ depends only on d.

To bound quotient of two local L^2-norms

$$\frac{\|\psi\|_{L^2(\Theta)}}{\|\psi\|_{L^2(B(x_0,\delta))}}$$

we need info on another such quotient!

$$\frac{\|\psi\|_{L^2(\Lambda)}}{\|\psi\|_{L^2(\Theta)}}$$

If an estimate on the latter is not provided a-priori, one might wonder, whether one is running in a vicious circle or an induction without induction anchor.

Where to get an a-priori bound?

One bound we can get by normalization: $||\psi||_{L^2(\Lambda)} = 1$

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On average we have for each of the L^d unit boxes

$$||\psi||_{L^2(\Lambda_1+j)} \sim L^{-d}$$

$$||\psi||_{L^2(\Lambda_1+j)} \geq \frac{1}{2} L^{-d}$$

Criterion introduced in [Rojas-Molina, Veselic 13]

The last inequality will hold for some boxed (dominating ones)
and not for others (weak boxes)

There are sufficiently many dominating boxes such that

$$\sum_{j,\text{dominating}} \int_{\Lambda_1(j)} |\psi|^2 \geq \frac{1}{2} \int_{\Lambda_L} |\psi|^2$$

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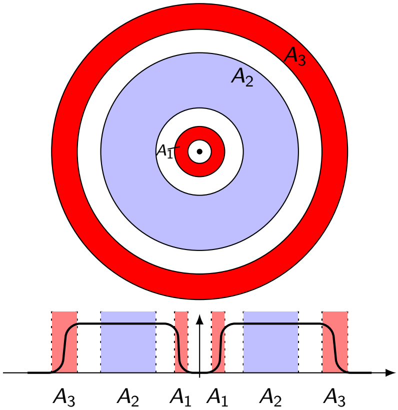
Uses

Lemma (A reverse Markov inequality)

Let $N, T \in \mathbb{N}$ and μ be a probability measure on $\overline{N} := \{1, \dots, N\}$. Set $A := \{n \in \overline{N} \mid \mu(n) \leq \frac{1}{T}\frac{1}{N}\}$. Then $\mu(A) \leq 1/T$.

Actually, dominating and weak boxes are defined in terms of an intermediate scale $1 \leq T = T(d) \ll L$.

Consequences/Applications of Carleman estimates: Three annuli inequality



Insert

$\chi \times \psi$ = cut-off × eigenfunction

in Carleman inequality
(e.g. [Bourgain, Kenig 05])
to get three annuli inequality

$$\alpha^3 \int_{A_2} w^{2-2\alpha} |\psi|^2 \lesssim \int_{A_1} w^{2-2\alpha} |\psi|^2 + \int_{A_3} w^{2-2\alpha} |\psi|^2.$$

profile of cut-off function χ

Application of 3 annuli inequality to equidistribution of an eigenfunction

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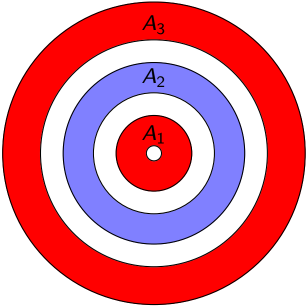

Choice of radii (widths and gaps)

$$r_1 = \delta/8, \quad r_2 = 1, \quad r_3 = 6\mathrm{e}\sqrt{d},$$
$$R_1 = \delta/4, \quad R_2 = 3\sqrt{d}, \quad R_3 = 9\mathrm{e}\sqrt{d},$$

$$A_i = B(R_i) \setminus B(r_i), i \in \{1, 2, 3\}$$

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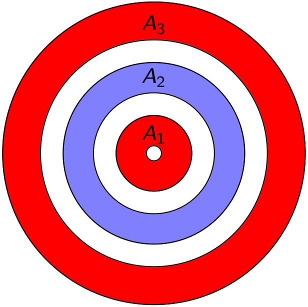

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Bounds on the weight function $|x|/(\mathrm{e}R_3) \le w(x) \le |x|/R_3$ imply

$$\alpha^3 \left(\frac{R_3}{R_2}\right)^{2\alpha-2} \int_{A_2} |\psi|^2 \lesssim \left(\frac{\mathrm{e}R_3}{r_1}\right)^{2\alpha-2} \int_{A_1} |\psi|^2 + \left(\frac{\mathrm{e}R_3}{r_3}\right)^{2\alpha-2} \int_{A_3} |\psi|^2$$

Application of 3 annuli inequality to equidistribution of an eigenfunction

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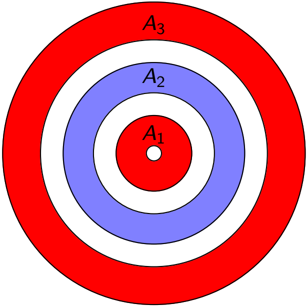

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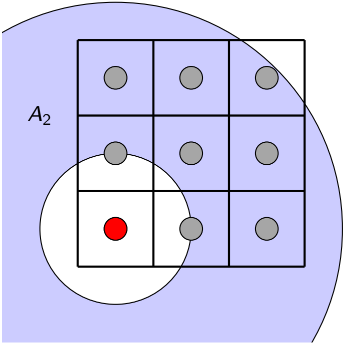
Apply this to the translated sets $A_i + x_j$ and sum up!

$$\alpha^3 \left(\frac{R_3}{R_2}\right)^{2\alpha-2} \sum_j \|\psi\|^2_{A_2+x_j} \lesssim \left(\frac{\mathrm{e}R_3}{r_1}\right)^{2\alpha-2} \sum_j \|\psi\|^2_{A_1+x_j} + \left(\frac{\mathrm{e}R_3}{r_3}\right)^{2\alpha-2} \sum_j \|\psi\|^2_{A_3+x_j}$$



Covering argument gives

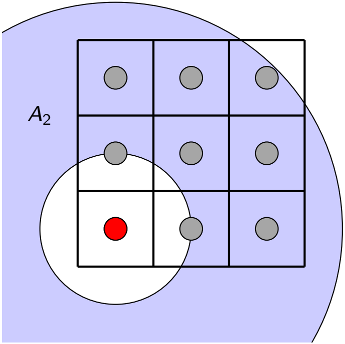
$$\sum_j \|\psi\|^2_{A_2+x_j} \geq \|\psi\|^2_{\Lambda}$$

$$\sum_j \|\psi\|^2_{A_1+x_j} \leq \|\psi\|^2_{S_\delta}$$

$$\sum_j \|\psi\|^2_{A_3+x_j} \leq K_d \|\psi\|^2_{\Lambda}$$

K_d combinatorial factor

$$\alpha^3 \left(\frac{R_3}{R_2}\right)^{2\alpha-2} \sum_j \|\psi\|^2_{A_2+x_j} \lesssim \left(\frac{\mathrm{e}R_3}{r_1}\right)^{2\alpha-2} \sum_j \|\psi\|^2_{A_1+x_j} + \left(\frac{\mathrm{e}R_3}{r_3}\right)^{2\alpha-2} \sum_j \|\psi\|^2_{A_3+x_j}$$



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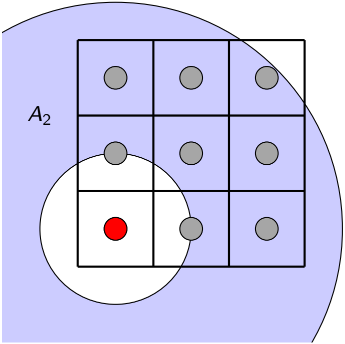
$$\sum_j \|\psi\|^2_{A_1+x_j} \leq \|\psi\|^2_{S_\delta}$$

$$\sum_j \|\psi\|^2_{A_3+x_j} \leq K_d \|\psi\|^2_\Lambda$$

K_d combinatorial factor

$$\alpha^3 \left(\frac{R_3}{R_2}\right)^{2\alpha-2} \|\psi\|^2_\Lambda \lesssim \left(\frac{\mathrm{e}R_3}{r_1}\right)^{2\alpha-2} \|\psi\|^2_{S_\delta} + \left(\frac{\mathrm{e}R_3}{r_3}\right)^{2\alpha-2} K_d \|\psi\|^2_\Lambda.$$

$$\alpha^3\left(\frac{R_3}{R_2}\right)^{2\alpha-2}\sum_j\|\psi\|^2_{A_2+x_j}\lesssim\left(\frac{\mathrm{e}R_3}{r_1}\right)^{2\alpha-2}\sum_j\|\psi\|^2_{A_1+x_j}+\left(\frac{\mathrm{e}R_3}{r_3}\right)^{2\alpha-2}\sum_j\|\psi\|^2_{A_3+x_j}$$



Covering argument gives

$$\sum_j\|\psi\|^2_{A_2+x_j}\geq\|\psi\|^2_\Lambda$$

$$\sum_j\|\psi\|^2_{A_1+x_j}\leq\|\psi\|^2_{S_\delta}$$

$$\sum_j\|\psi\|^2_{A_3+x_j}\leq K_d\|\psi\|^2_\Lambda$$

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$$\alpha^3\left(\frac{R_3}{R_2}\right)^{2\alpha-2}\|\psi\|_\Lambda^2 \lesssim \left(\frac{\mathrm{e}R_3}{r_1}\right)^{2\alpha-2}\|\psi\|_{S_\delta}^2 + \left(\frac{\mathrm{e}R_3}{r_3}\right)^{2\alpha-2} K_d\|\psi\|_\Lambda^2.$$

Choice of gaps between annuli ensures

$$\left(\frac{R_3}{R_2}\right)^{2\alpha-2} \gg \left(\frac{\mathrm{e}R_3}{r_3}\right)^{2\alpha-2}$$

thus for large α

$$\|\psi\|_\Lambda^2 \lesssim \left(\frac{\mathrm{e}R_2}{r_1}\right)^{2\alpha-2}\|\psi\|_{S_\delta}^2$$

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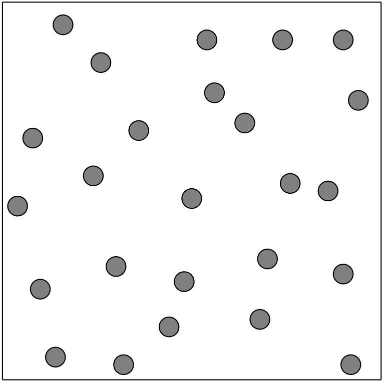
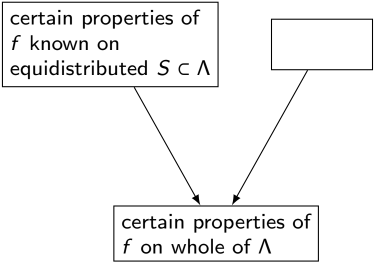
$$\|\psi\|_\Lambda^2 \lesssim \left(\frac{\mathrm{e}R_2}{r_1}\right)^{2\alpha-2} \|\psi\|_{S_\delta}^2 = \left(\frac{24\mathrm{e}\sqrt{d}}{\delta}\right)^{2\alpha-2} \|\psi\|_{S_\delta}^2$$

Uniform Uncertainty Principles and restricted isometry constants

Retrieval of global properties from local data

Let $\Lambda \subset \mathbb{R}^d$ be a region in space, $S \subset \Lambda$ a subset, and $f\colon \Lambda \to \mathbb{R}$.





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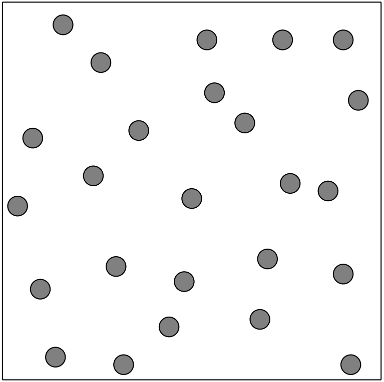

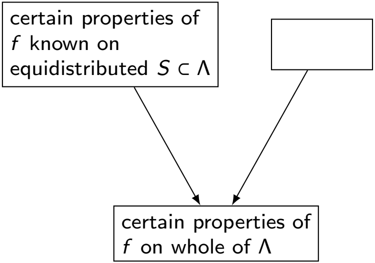

What can one say about certain properties of $f\colon \Lambda \to \mathbb{R}$ given certain properties of $f\,|_S\colon S \to \mathbb{R}$?

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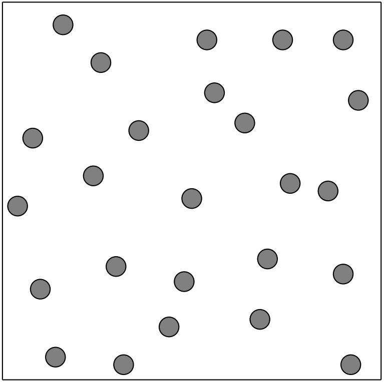

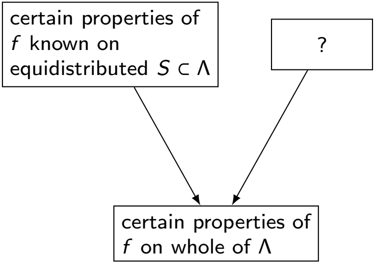

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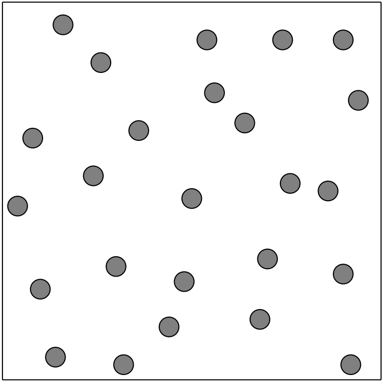

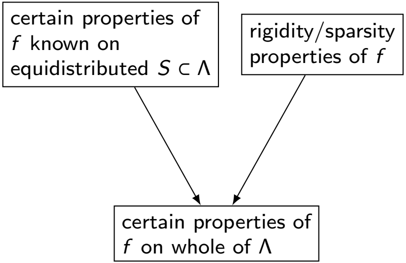

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Uniform Uncertainty Principles and restricted isometry constants

Definition

Let $D, m, s \in \mathbb{N}$, matrix $B\colon \mathbb{R}^D \to \mathbb{R}^m$, and $s \leq D$, (typically: $D \gg m$). If

$$(1-\delta_s)\|y\|^2 \leq \|By\|^2 \leq (1+\delta_s)\|y\|^2 \tag{1}$$

for all $y \in \mathbb{R}^D$ with $\sharp \operatorname{supp} y \leq s$, then δ_s is called a *restricted isometry constant* (for s and B).

Lemma [Candes, Romberg, Tao '06]

Let D, m, s and B be as above with $1 - \delta_s > 0$. Let $x \in \mathbb{R}^D$ with $\sharp \operatorname{supp} x \leq s$ and $f := Bx$. Then the unique minimizer of

$$(P_0) \qquad \min_{y \in \mathbb{R}^D;\, By = f} \sharp \operatorname{supp} y$$

equals x.

Program (P_0) has high complexity and needs to be replaced by a convex program.

Uniform Uncertainty Principles and restricted isometry constants

Theorem [Candes, Romberg, Tao '06]

Let $D, m, s \in \mathbb{N}$, $\epsilon \geq 0$, and $B\colon \mathbb{R}^D \to \mathbb{R}^m$ such that $\delta_{3s} + 3\delta_{4s} < 2$. Assume that $x \in \mathbb{R}^D$ and $e \in \mathbb{R}^m$ satisfy $\sharp \operatorname{supp} x \leq s$ and $\|e\|_2 \leq \epsilon$. Set $f := Bx + e$. Then the solution ξ of the convex optimization problem

$$(P_2) \qquad \min_{y \in \mathbb{C}^D; \|By - f\|_2 \leq \epsilon} \|y\|_1$$

obeys

$$\|\xi - x\|_2 \leq C(\delta_{4s}) \cdot \epsilon$$

Condition $\delta_{3s} + 3\delta_{4s} < 2$ has been relaxed since, see [Foucart, Rauhut 13]

It turns out that this result is stable under small perturbations of x where the support condition is violated.

Truncated vectors

For $x \in \mathbb{R}^D$ and $s < D$ denote by $x_s \in \mathbb{R}^D$ the vector where the $D - s$ coefficients closest to zero have been set equal to zero.

Stability

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Theorem [Candes, Romberg, Tao '06]

Let $D, m, s \in \mathbb{N}$, $\epsilon \geq 0$, $x \in \mathbb{R}^D$, $e \in \mathbb{R}^m$, and $B \colon \mathbb{R}^D \to \mathbb{R}^n$ such that $\delta_{3s} + 3\delta_{4s} < 2$. Assume that $\|e\|_2 \leq \epsilon$. Set $f := Bx + e$. Then the solution ξ of

$$(P_2) \qquad \min_{y \in \mathbb{C}^D; \|By - f\|_2 \leq \epsilon} \|y\|_1$$

obeys

$$\|\xi - x\|_2 \leq C_1 \cdot \epsilon + C_2 \frac{\|x - x_s\|_1}{\sqrt{s}}$$

where C_1 and C_2 depend only on δ_{4s}.

Matrix ensembles with uniform Uncertainty Principles

No efficient deterministic way known to construct matrices satisfying uniform Uncertainty Principle of arbitrary size, but probabilistic ones:

Definition

Let $D, m, s \in \mathbb{N}$, $B\colon \mathbb{R}^D \to \mathbb{R}^m$, and $s \le D$, (typically: $D \gg n$). If

$$(1 - \delta_s)\|y\|^2 \le \|By\|^2 \le (1 + \delta_s)\|y\|^2 \tag{2}$$

for all $y \in \mathbb{R}^D$ with $\operatorname{supp} y \le s$, then δ_s is called a *restricted isometry constant* (for s and B).

Fourier ensemble

Let F be the $D \times D$ discrete Fourier transform matrix. Select m rows randomly, and normalise each column, to obtain B. Then uniform Uncertainty Principle holds with probability very close to one if

$$s \le const. \frac{m}{(\log D)^6}$$

Analogs in ∞ dimensions?: Logvinenko-Sereda Theorem

Let $S \subset \mathbb{R}$ be measurable and $\gamma, a > 0$ such that, for all intervals $I \subset \mathbb{R}$ of length a

$$|S \cap I| \geq \gamma \cdot a \qquad a \text{ is a scale, } \gamma \text{ a density.}$$

Then S is called (γ, a)-thick.

Theorem [Logvinenko-Sereda '74]

Let $p \in [1, \infty]$ and $J \subset \mathbb{R}$ be an interval of length $b > 0$. Let $f \in L^p(\mathbb{R})$ with $\hat{f}$ supported in J. Then

$$C(ab, \gamma) \, \|f\|_p \leq \|\chi_S f\|_p$$

Logvinenko-Sereda: $C(ab, \gamma) = \exp\Big(- c \frac{1+ab}{\gamma} \Big)$

Kovrijkine'01: $C(ab, \gamma) = \Big(\frac{\gamma}{c} \Big)^{c(1+ab)}$

Kovrijkine-Logvinenko-Sereda Theorem [Kovrijkine '01]

Let $p \in [1, \infty]$ and $J_1, \ldots, J_n \subset \mathbb{R}$ be intervals of length $b > 0$. Let $f \in L^p(\mathbb{R})$ with $\hat{f}$ supported in $J_1 \cup \ldots \cup J_n$. Then

$$C \, \|f\|_p \leq \|\chi_S f\|_p, \qquad C = \Big(\frac{\gamma}{c} \Big)^{ab \left(\frac{c}{\gamma} \right)^n + n}.$$

Reformulation as Uniform Uncertainty Principle

Reformulation of Kovrijkine's Theorem

Fix $\gamma, a, b > 0, n \in \mathbb{N}$. Let $B\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the multiplication operator with the characteristic function of a (γ, a)-thick set. For an interval J of length b set $\hat{L}(J) := \{f \in L^2(\mathbb{R}) \,|\, \operatorname{supp} \hat{f} \subset J\}$. While B is not injective, we have

$$\left(\frac{\gamma}{c}\right)^{ab(c/\gamma)^n+n} \|\psi\|_2 \leq \|B\,\psi\|_2 \leq \|\psi\|_2 \tag{3}$$

for all $\psi \in \bigcup \mathop{\dotplus}_{k=1}^{n} \hat{L}(J_l)$, where the union runs over all n-tuples $J_1, \ldots, J_n \subset \mathbb{R}$ of intervals of length b each.

ψ is 'sparse' in Fourier representation.

None of the subspaces $\hat{L}(J_k)$ has finite dimension, but they are all unitarily equivalent. In particular, the constant c in (3) does not depend on the positions of the intervals J_k.

Kovrijkine obtained also analogous results in higher dimensions $d \in \mathbb{N}$

Finite interval/torus: Finite dimensional subspaces

Configuration space $\mathbb{T}_L^d$ of size $L > 0$ and dimension $d \in \mathbb{N}$.

Theorem [Egidi & Ves. '16]

Let $f \in L^p(\mathbb{T}_L^d)$ with $p \in [1, \infty]$ such that $\operatorname{supp} \hat{f} \subset J$, where J is a box in $\mathbb{R}^d$ with side lengths $b_1, \dots, b_d$, set $b = (b_1, \dots, b_d)$. Let $S \subset \mathbb{R}^d$ be a (γ, a)-thick set with $a = (a_1, \dots, a_d)$ such that $0 < a_j \le 2\pi L$ for all $j = 1, \dots, d$. Then,

$$\left(\frac{\gamma}{c_1^d}\right)^{c_2 a\cdot b + \frac{4d+1}{p}} \|f\|_{L^p(\mathbb{T}_L^d)} \le \|f\|_{L^p(S\cap\mathbb{T}_L^d)} \le \|f\|_{L^p(\mathbb{T}_L^d)}, \tag{4}$$

where c_1, c_2 are universal constants.

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Version for several boxes

Let $f \in L^p(\mathbb{T}_L^d)$ with $p \in [1, \infty]$. Assume that $\operatorname{supp} \hat{f} \subset \bigcup_{l=1}^n J_l$, where J_l are boxes in $\mathbb{R}^d$ with sides of length $b_1, \ldots, b_d$. Let $E \subset \mathbb{R}^d$ be a (γ, a)-thick set with $a = (a_1, \ldots, a_n)$ such that $0 < a_j \leq 2\pi L$ for all $j = 1, \ldots, d$. Then,

$$\left(\frac{\gamma}{\tilde{c}_1}\right)^{\left(\frac{\tilde{c}_2^d}{\gamma}\right)^n \sum_{j=1}^d a_j b_j + n} \|f\|_{L^p(\mathbb{T}_L^d)} \leq \|f\|_{L^p(E \cap \mathbb{T}_L^d)}, \tag{5}$$

where $\tilde{c}_1$ and $\tilde{c}_2$ are universal constants.

In particular the estimates are scale-free in the sense that they do not depend on the size L of the torus.

Relation to compressed sensing

We obtained a restricted isometry property or Uniform Uncertainty Principle in in $L^2(\mathbb{T}_L)$, however the restricted isometry constant:

$$\delta = 1 - \left(\frac{\gamma}{\tilde{c}_1}\right)^{\left(\frac{\tilde{c}_2^d}{\gamma}\right)^n \sum_{j=1}^d a_j b_j + n}$$

is very close to one and not to zero $\Longrightarrow$ to large to apply method of Candes and Tao.

However, we expected this. Otherwise the result would be to good to be true: extension to infinite dimensional space without probabilistic error.

Are there at all instances where the reconstruction above mentioned method works in infinite dimensions?

Yes, under additional assumptions, see e.g.

- [Donoho & Logan] Set E is very thick with density γ close to one.
- [Jean-Pierre Kahane] Fourier-coefficients satisfy lacunary gap condition

Future objectives:

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Thank you for your attention!