LONG-TIME ASYMPTOTICS FOR KDV AND TODA EQUATIONS WITH STEPLELIKE INITIAL PROFILE

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The Cauchy problem

We are interested in the long-time asymptotic behaviour of the solution of the Korteweg - de Vries equation

\[ q_t(x, t) = 6q(x, t)q_x(x, t) - q_{xxx}(x, t), \quad (x, t) \in \mathbb{R} \times \mathbb{R}_+, \]

with steplike initial profile \( q(x, 0) = q_0(x) \in C^n(\mathbb{R}) \):

\[
\begin{cases}
q_0(x) \to c_+, & \text{as } x \to +\infty, \\
q_0(x) \to c_-, & \text{as } x \to -\infty,
\end{cases}
\]

with

\[
\int_{\mathbb{R}_+} (1 + |x|^m)|q_0(\pm x) - c_\pm| \, dx < \infty, \quad m \geq 1.
\]
Without loss generality one can put $c_+ = 0$. If $c_- = -c^2$, $c \in \mathbb{R}$, we say about the shock problem for the KdV equation. The case $c_- = c^2$ is known as the rarefaction problem.

Initial data

$$q_0(x) = \begin{cases} 
0, & x \geq 0, \\
\pm c^2, & x < 0,
\end{cases}$$

are called pure step ones.

Both shock and rarefaction problem are studied in the regime:

$$x \to \infty; \quad t \to +\infty; \quad \frac{x}{t} = O(1) \text{ as } t \to \infty.$$
Shock problem

Well understood on a physical level of rigor for pure step initial data (Gurevich/Pitaevskii '73, Bikbaev '89, Sharipov/Novokshenov '78, Leach/Needham '08):

- in the domain $x < -6c^2t$ the solution is asymptotically close to the background constant $-c^2$;
- in the domain $-6c^2t < x < 4c^2t$ the solution is described by the modulated elliptic wave (Whitham’s method);

Inverse scattering transform (IST), steplike initial data: Ermakova ’81:

- in the domain $4c^2t < x$ the solution is asymptotically close to the sum of solitons:

$$q(x, t) = -2 \sum_{j=1}^{N} \frac{\kappa_j^2}{\cosh^2(\kappa_j x - 4\kappa_j^3 t - p_j)} + O(t^{-1}).$$
Asymptotical solitons

Khruslov, ’76

In the domain $4c^2 t \geq x > 4c^2 t - (2c)^{-1} \ln t^{M+1}$ as $t \to +\infty$:

$$q(x, t) = \sum_{k=1}^{[M+1/2]} \frac{-2c^2}{\cosh^2\{c x - 4c^3 t + \frac{1}{2} \ln t^{2n-1/2} + \phi_k\}} + O(t^{-1/2+\varepsilon}),$$

where $\varepsilon > 0$ is arbitrary small and the phases $\phi_k$ are determined by the initial scattering data.
Figure: $t = 10$, $q(x, 0) = \frac{1}{2}(\text{erf}(x) - 1) - 5\text{sech}(x - 1)$. 
Rarefaction problem

Well understood on the physical level of rigor for pure step initial data (Leach/Needham ’14, Zakharov/Manakov/Novikov/Pitaevskii ’80):

\[ q(x, t) = c^2 + o(1) \text{ in the domain } x < -6c^2t; \]
\[ q(x, t) = -\frac{x}{6t} + o(1) \text{ in the domain } -6c^2t < x < 0; \]
\[ \text{For } x > 0 \text{ the solution is asymptotically close to the sum of solitons.} \]

**Figure:** \( t = 1.5, \quad q_0(x) = \frac{1}{2}(1 - \text{erf}(x)) - 4 \text{ sech}(x - 1). \)
Dispersion: rapidly decaying initial condition

Let us take the linear part of the KdV equation

\[ q_t = -q_{xxx} + 6qq_x, \quad q(x, t = 0) = q_0(x) \rightarrow 0, \ x \rightarrow \pm \infty, \]

then the equation can be solved using the Fourier transform

\[ q(x, t) = \int_{-\infty}^{\infty} \hat{q}_0(k) e^{i(kx + k^3 t)} \, dk = \int_{-\infty}^{\infty} \hat{q}_0(k) e^{t \Phi(k)} \, dk, \]

\[ \hat{q}_0(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} q_0(x) e^{-ikx} \, dx. \]

where the phase is given by

\[ \Phi(k) = i(ku + k^3), \quad u = \frac{x}{t}. \]

We are interested in the asymptotics as \( t \rightarrow \infty \) keeping the velocity \( u \) fixed. To this end we need to look at the stationary phase points

\[ \Phi'(k) = 0 \quad \Rightarrow \quad k = \pm k_0, \ k_0 = \pm \sqrt{-\frac{u}{3}}. \]
Steepest descent

Assuming \( \hat{q}_0(k) \) admits an analytic continuation to some strip \( |\text{Im}(k)| < \epsilon \) we can deform the integration contour \( \mathbb{R} \to \Sigma \) into regions of the complex plane where the exponent \( e^{t\Phi(k)} \) decays exponentially as \( t \to \infty \). For \( u = \frac{x}{t} > 0 \) there are no stationary phase points on \( \mathbb{R} \) and we just shift the path of integration a bit up to obtain exponential decay:

\[
\begin{align*}
\text{Re}(\Phi) > 0 & \quad \text{Re}(\Phi) < 0 \\
\text{Re}(\Phi) < 0 & \quad \text{Re}(\Phi) > 0
\end{align*}
\]
Assuming \( \hat{q}_0(k) \) admits an analytic continuation to some strip \(|\text{Im}(k)| < \epsilon\) we can deform the integration contour \( \mathbb{R} \to \Sigma \) into regions of the complex plane where the exponent \( e^{t\Phi(k)} \) decays exponentially as \( t \to \infty \). For \( u = \frac{x}{t} > 0 \) there are no stationary phase points on \( \mathbb{R} \) and we just shift the path of integration a bit up to obtain exponential decay:
For \( u = \frac{x}{t} < 0 \) there are two stationary phase points \( \pm k_0 \in \mathbb{R} \) and we need to pass through them:

\[
\begin{align*}
\text{Re}(\Phi) &> 0 \\
\text{Re}(\Phi) &< 0 \\
\end{align*}
\]

\[
\begin{align*}
\text{Re}(\Phi) &< 0 \\
\text{Re}(\Phi) &> 0 \\
\end{align*}
\]
For \( u = \frac{x}{t} < 0 \) there are two stationary phase points \( \pm k_0 \in \mathbb{R} \) and we need to pass through them:

There is now a contribution of \( O(t^{-1/2}) \) from the two stationary phase points which can be computed by expanding around these points and explicitly evaluating the resulting integrals.
Solutions of the linearized KdV equation

Hence a typical solution of the linearized KdV equation with decaying initial condition splits into a slowly decaying dispersive tail to the left and a fast decaying part to the right. Below a numerically computed solution for \( q_0(x) = e^{-x^2} \) is shown:
Now let us look at the **nonlinear** part of the KdV equation

\[ q_t = -q_{xxx} + 6qq_x, \quad q(x, t = 0) = q_0(x). \]

This is **inviscid Burgers’ equation** which can be solved using the method of characteristics. The wave packets will steepen until they break:

![Wave packets steepening and breaking](image)

It turns out that for KdV both effects balance each other and give rise to traveling wave solutions - solitons:
Rapidly decaying initial condition, numerics

Figure: \( t = 5, \ q(x, 0) = \text{sech}(x + 3) - 5\text{sech}(x - 1). \)

Long-time asymptotical analysis: Ablowitz/Newell '73, Shabat '73, Tanaka '73, Manakov '74, Ablowitz/Segur '77, Buslaev '81, Buslaev/Sukhanov '86, Deift/Venakides/Zhou '94, Grunert/Teschl '10.
The KdV equation is equivalent to the Lax equation

\[ \frac{dL(t)}{dt} = L(t)A(t) - A(t)L(t), \]

where

\[ L(t) = -\frac{\partial^2}{\partial x^2} + q(x, t), \]

\[ A(t) = -4\frac{\partial^3}{\partial x^3} + 3 \left( q(x, t) \frac{\partial}{\partial x} + \frac{\partial}{\partial x} q(x, t) \right) \]
The initial value problem for the KdV equation can be solved via the inverse scattering transform:

\[ q(x, 0) \] to \[ q(x, t) \]

The long-time asymptotics can then be found via a nonlinear steepest descent analysis.
Consider the KdV equation with steplike initial data $q_0(x)$, and let the solution of the initial value problem satisfies

$$\int_0^\infty |x||q(x, t)| + |q(-x, t) \pm c^2|)dx < \infty, \quad \forall t \in \mathbb{R}.$$
Consider the KdV equation with steplike initial data $q_0(x)$, and let the solution of the initial value problem satisfies

$$\int_0^\infty |x|(|q(x, t)| + |q(-x, t) \pm c^2|)dx < \infty, \quad \forall t \in \mathbb{R}.$$ 


**Rarefaction problem:** $q_0(x) \to c^2$ as $x \to -\infty$.

Spectral picture for the Schrödinger operator

$$L(t) = \partial_x^2 + q(x, t):$$

\[
\begin{array}{cccccccc}
\bullet & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-\kappa_1^2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

$\Sigma^{(1)}$ $\Sigma^{(2)}$

\[
\begin{array}{cccccccc}
\bullet & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-\kappa_p^2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

0 $c^2$
The Schrödinger equation $L(t)\phi = k^2\phi$ has the Jost solutions:

$$
\lim_{x \to +\infty} e^{-ikx}\phi(k, x, t) = \lim_{x \to -\infty} e^{ik_1x}\phi_1(k, x, t) = 1, \quad \text{as } k \in \mathbb{C}^+.
$$

Here $k_1 = \sqrt{k^2 - c^2}$. They can be represented via the transformation operators:

$$
\phi(k, x, t) = e^{ikx} + \int_x^{+\infty} K(x, y, t)e^{iky} dy,
$$

$$
\phi_1(k, x, t) = e^{-ik_1x} + \int_{-\infty}^x K_1(x, y, t)e^{-ik_1y} dy,
$$

where

$$
|K(x, y, t)| \leq C \int_{\frac{x+y}{2}}^{\infty} |q(z, t)| dz, \quad K_1(x, y, t) | \leq C \int_{-\infty}^{\frac{x+y}{2}} |q(z, t) - c^2| dz.
$$

In particular, $\phi_1(k, x, t) \in \mathbb{R}$ as $k \in [-c, c]$. 
The Jost solutions are connected by the scattering relations

\[ T(k, t)\phi_1(k, x, t) = \phi(k, x, t) + R(k, t)\phi(k, x, t), \quad k \in \mathbb{R}, \]

\[ T_1(k, t)\phi(k, x, t) = \phi_1(k, x, t) + R_1(k, t)\phi_1(k, x, t), \quad k \in \mathbb{R}\setminus[-c, c], \]

- \( T(k, t) \) and \( R(k, t) \) are the right transmission and reflection coefficients.
- The following representations are valid:

\[ T(k, t) = \frac{2ik}{\langle \phi_1, \phi \rangle}, \quad R(k, t) = -\frac{\langle \phi_1, \bar{\phi} \rangle}{\langle \phi_1, \phi \rangle}, \]

where \( \langle f, g \rangle = fg_x - gf_x \) is the Wronskian.
Scattering matrix

- The Jost solutions are connected by the scattering relations
  \[ T(k, t)\phi_1(k, x, t) = \phi(k, x, t) + R(k, t)\phi(k, x, t), \quad k \in \mathbb{R}, \]
  \[ T_1(k, t)\phi(k, x, t) = \phi_1(k, x, t) + R_1(k, t)\phi_1(k, x, t), \quad k \in \mathbb{R}\setminus[-c, c], \]

- \( T(k, t) \) and \( R(k, t) \) are the right transmission and reflection coefficients.

- The following representations are valid:
  \[ T(k, t) = \frac{2ik}{\langle \phi_1, \phi \rangle}, \quad R(k, t) = -\frac{\langle \phi_1, \bar{\phi} \rangle}{\langle \phi_1, \phi \rangle}, \]

where \( \langle f, g \rangle = fg_x - gf_x \) is the Wronskian.

In particular, \(|R(k, t)| = 1\) as \( k \in [-c, c] \). \( R(0, t) = -1 \) corresponds to the non-resonant case, \( R(0, t) = 1 \) is the resonant case.
The solution $\phi(i\kappa_j, x, t) \in \mathbb{R}$ is an eigenfunction of $L(t)$. The value

$$
\gamma_j(t) = \left( \int_{\mathbb{R}} \phi^2(i\kappa_j, x, t) dx \right)^{-1},
$$

is called the right normalizing constant.

The solution $q(x, t)$ can be uniquely restored from the right scattering data

$$
S(L(t)) := \{ R(k, t), \ k \in \mathbb{R}; \ -\kappa_j^2, \ \gamma_j(t) > 0, \ j = 1, \ldots, N \}.
$$

The time evolution of the scattering data can be computed explicitly from the Lax equation:

$$
R(k, t) = R(k)e^{8ik^3t}, \quad T(k, t) = T(k)e^{4ik^3t - 4ik_1^3t + 6ic k_1 t},
$$

$$
\gamma_j(t) = \gamma_j e^{8\kappa_j^3t}.
$$

Here $R(k) = R(k, 0), \ \gamma_j = \gamma_j(0)$. 
In \( \mathbb{C}^+ \) introduce the vector function

\[
m(k, x, t) = (T(k, t)\phi_1(k, x, t)e^{ikx}, \phi(k, x, t)e^{-ikx}).
\]

Then \( q(x, t) \) can be read off from \( m \) via

\[
m(k) = (1 \ 1) - \frac{1}{2ik} \int_x^{+\infty} q(y, t)dy (-1 \ 1) + O \left( \frac{1}{k^2} \right).
\]

Define \( m(k) \) in \( \mathbb{C}^- \) by use of the symmetry condition:

\[
m(k) = m(-k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]
Theorem (no discrete spectrum)

The function \( m(k) \) is the unique solution of the following Riemann-Hilbert problem: to find a holomorphic in \( \mathbb{C} \setminus \mathbb{R} \) function \( m(k) \), which has continuous limits \( m_{\pm}(k) = m(k \pm i0), \ k \in \mathbb{R} \), and satisfies:

1. **the jump condition** \( m_+(k) = m_-(k)v(k) \), where

\[
v(k) := v(k, x, t) = \begin{pmatrix} 1 - |R(k)|^2 & -\overline{R(k)}e^{-2t\Phi(k)} \\ R(k)e^{2t\Phi(k)} & 1 \end{pmatrix}, \quad k \in \mathbb{R};
\]

2. **the symmetry condition**;

3. **the normalizing condition**

\[
m(k) = (1 \quad 1) + O(k^{-1}), \quad k \to \infty.
\]

Here **the phase function** is given by \( \Phi(k) = 4ik^3 + 12ik\xi \), where \( \xi = \frac{x}{12t} \).
Let $d(k)$ be a holomorphic function on $\mathbb{C} \setminus \Sigma$, where $\Sigma \subset \mathbb{R}$ is a symmetric with respect to the map $k \mapsto -k$ contour. If $d(-k) = d^{-1}(k)$ and $d(k) \to 1$, as $k \to \infty$, then the transformation

$$\tilde{m}(k) = m(k) \begin{pmatrix} d^{-1}(k) & 0 \\ 0 & d(k) \end{pmatrix}$$

preserves the symmetry and the normalization conditions.
Admissible transformations for RH problem

- Let $d(k)$ be a holomorphic function on $\mathbb{C} \setminus \Sigma$, where $\Sigma \subset \mathbb{R}$ is a symmetric with respect to the map $k \mapsto -k$ contour. If $d(-k) = d^{-1}(k)$ and $d(k) \to 1$, as $k \to \infty$, then the transformation

$$\tilde{m}(k) = m(k) \begin{pmatrix} d^{-1}(k) & 0 \\ 0 & d(k) \end{pmatrix}$$

preserves the symmetry and the normalization conditions.

- Let $\Omega^U \subset \mathbb{C}^+$ be a domain and let $\Omega^L \subset \mathbb{C}^-$ be symmetric with respect to the maps $k \mapsto -k$ and $k \mapsto \bar{k}$. The transformation

$$\tilde{m}(k) = m(k) \begin{cases} B_{U,L}(k), & k \in \Omega^{U,L}, \\
I, & \text{else}, \end{cases}$$

is admissible if

$$B_U(k) = \begin{pmatrix} 1 & a(k) \\ 0 & 1 \end{pmatrix}, \quad B_L(k) = \begin{pmatrix} 1 & 0 \\ a(-k) & 1 \end{pmatrix},$$

or vice versa, is admissible.
For \( k \in \mathbb{R} \setminus [-\sqrt{-\xi}, \sqrt{-\xi}] \), we factorize the jump matrix:

\[
\nu(k) = \begin{pmatrix} 1 & -R(k)e^{-2t\Phi(k)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ R(k)e^{2t\Phi(k)} & 1 \end{pmatrix} = B_L \cdot B_U^{-1},
\]

\[
\text{Re } \Phi = 0
\]

Figure: Signature table for \( \text{Re } \Phi(k) \) as \( \xi < 0 \).
The lenses mechanizm

Put

\[ m^{(1)}(k) = m(k) \begin{cases} B_U(k), & k \in \Omega^U; \\ B_L(k), & k \in \Omega^L; \end{cases} \]

Then \( m^{(1)}_+(k) = m^{(1)}_-(k) \nu^{(1)}(k) \), where for \( t \to \infty \)

\[ \nu^{(1)}(k) = \begin{cases} \nu(k), & \left[ -\sqrt{-\xi}, \sqrt{-\xi} \right], \\ \mathbb{I} + o(1), & k \in C^U \cup C^L. \end{cases} \]
"Lower - upper" factorization

On \( k \in [-\sqrt{-\xi}, \sqrt{-\xi}] \) we solve an auxiliary scalar RH problem

\[
d_+(k) = d_-(k)(1 - |R(k)|^2), \quad d(-k) = d^{-1}(k), \quad d(k) \to 1, \text{ as } k \to \infty,
\]

and put \( m^{(2)}(k) = m^{(1)}(k)d(k)^{\sigma_3} \), where \( \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \), then

\[
v^{(2)}(k) = \begin{pmatrix} 1 & 0 \\ \frac{d_-(k)^2R(k)e^{2t\Phi(k)}}{1-|R^2(k)|} & 1 \end{pmatrix} \begin{pmatrix} 0 & -\frac{d_+(k)^2R(k)e^{-2t\Phi(k)}}{1-|R^2(k)|} \\ 1 & 1 \end{pmatrix} = \tilde{B}_L\tilde{B}_U^{-1}.
\]
In the rarefaction wave case $|R(k)| = 1$ as $k \in [-c; c]$.

We study the region between the leading and the rear wave fronts,

$$\xi \in \left( -\frac{c^2}{2}, 0 \right).$$
Rarefaction problem, the ”$g$-function” approach

In the rarefaction wave case $|R(k)| = 1$ as $k \in [-c; c]$. We study the region between the leading and the rear wave fronts,

$$
\xi \in \left(-\frac{c^2}{2}, 0\right).
$$

Put $a(\xi) = \sqrt{-2\xi}$ and

$$
g(k) = g(k; \xi) = 4i(k^2 - a^2(\xi))\sqrt{k^2 - a^2(\xi)}.
$$

Then $a(\xi) \in [0, c]$ and

$$
\Phi(k) - g(k) = \frac{12\xi^2}{2ik}(1 + O(k^{-1})), \quad k \to \infty.
$$
\[ \text{Figure: Signature tables for } \Re g \text{ and } \Re \Phi \text{ (dashed line)} \]
STEP 1. Put
\[ m^{(1)}(k) = m(k)e^{-t(\Phi(k)-g(k))}\sigma_3, \]
where \( \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) - is the Pauli matrix. Then \( m^{(1)}_+ = m^{(1)}_\times \nu^{(1)} \), where
\[
\nu^{(1)}(k) = \begin{cases} 
\begin{pmatrix} 0 & -\overline{R(k)} \\
R(k) & e^{-2tg_+(k)} \end{pmatrix}, & k \in [-a, a]; \\
\begin{pmatrix} 1 - |R(k)|^2 & -\overline{R(k)}e^{-2tg(k)} \\
R(k)e^{2tg(k)} & 1 \end{pmatrix}, & k \in \mathbb{R} \setminus [-a, a].
\end{cases}
\]

STEP 2. We perform the upper-lower factorization for \( \nu^{(1)}(k) \) as \( k \in \mathbb{R} \setminus [-a; a] \) as above, and apply the lenses mechanism. We get the RH problem for \( m^{(2)}(k) \) with
\[
\nu^{(2)}(k) = \begin{cases} 
\begin{pmatrix} 0 & -\overline{R(k)} \\
R(k) & o(1) \end{pmatrix}, & k \in [-a, a]; \\
\mathbb{I} + o(1), & k \in C^L \cup C^U.
\end{cases}
\]
The scalar conjugation problem:

To find a holomorphic in \( \mathbb{C} \setminus [−a, a] \) function \( d(k) \), satisfying the jump

\[
d_+(k)d_-(k) = R^{-1}(0)R(k), \quad k \in [−a, a],
\]

the symmetry \( d(−k) = d^{-1}(k) \), and normalisation \( d(k) \to 1 \), as \( k \to \infty \).
The scalar conjugation problem:

To find a holomorphic in $\mathbb{C} \setminus [-a, a]$ function $d(k)$, satisfying the jump

$$d_+(k)d_-(k) = R^{-1}(0)R(k), \quad k \in [-a, a],$$

the symmetry $d(-k) = d^{-1}(k)$, and normalisation $d(k) \to 1$, as $k \to \infty$.

STEP 3. Setting $m^{(3)}(k) = m^{(2)}(k)d(k)^{-\sigma_3}$, we get

$$v^{(3)}(k) = \begin{cases} 
\begin{pmatrix}
0 & -R(0) \\
R(0) & \frac{d_+(k)}{d_-(k)}e^{-2\tan(k_+)}
\end{pmatrix}, & k \in [-a, a]; \\
1 & 0 \\
d(k)^{-2}R(k)e^{2\tan(k)} & 1,
\end{cases}$$

$$v^{(3)}(k) = \begin{cases} 
\begin{pmatrix}
1 & -d(k)^2R(-k)e^{-2\tan(k)} \\
0 & 1
\end{pmatrix}, & k \in \mathcal{C}^L.
\end{cases}$$

Here $R(0) = \pm 1$. 
The model problem

To find a holomorphic in $\mathbb{C} \setminus [-a, a]$ vector function $m^{\text{mod}}(k)$, satisfying the jump

$$m^{\text{mod}}_+(k) = m^{\text{mod}}_-(k) \begin{pmatrix} 0 & -R(0) \\ R(0) & 0 \end{pmatrix}, \quad k \in [-a, a],$$

and the symmetry and normalization conditions:

$$m^{\text{mod}}(k) = m^{\text{mod}}(-k)\sigma_1; \quad m^{\text{mod}}(k) = (1, 1) + O(k^{-1}), \quad k \to \infty.$$

The matrix solution of the model problem:

$$M^{\text{mod}}(k) = \begin{pmatrix} \frac{\beta(k) + \beta^{-1}(k)}{2} & \frac{\beta(k) - \beta^{-1}(k)}{2i} \\ -\frac{\beta(k) - \beta^{-1}(k)}{2i} & \frac{\beta(k) + \beta^{-1}(k)}{2} \end{pmatrix}, \quad M^{\text{mod}}(k) \to \mathbb{I}, k \to \infty,$$

where

$$\beta(k) = \begin{cases} \sqrt[4]{\frac{k-a}{k+a}}, & \text{as } R(0) = -1 \text{ (non-resonant case)}; \\ \sqrt[4]{\frac{k+a}{k-a}}, & \text{as } R(0) = 1 \text{ (resonant case)}. \end{cases}$$
The solution of the model problem

The vector solution: \( m_{\text{mod}}(k) = (1, 1) M_{\text{mod}}(k) = \)

\[
= \frac{1}{2i} (\beta(k)(i - 1) + \beta^{-1}(k)(i + 1), \ \beta(k)(i + 1) + \beta^{-1}(k)(i - 1)).
\]

We prove that this solution approximates properly the solution of the initial RH problem for sufficiently large \( k \):

\[
m_1(k) = m_{1}^{(3)}(k) d(k) e^{t(\Phi(k) - g(k))} \sim m_{1}^{\text{mod}}(k) d(k) e^{t(\Phi(k) - g(k))}.
\]

Then from formula

\[
m(k) = (1 \ 1) - \frac{1}{2ik} \int_{x}^{+\infty} q(y, t) dy (-1 \ 1) + O \left( \frac{1}{k^2} \right)
\]

we compute

\[
q(x, t) = - \frac{\partial}{\partial x} \lim_{k \to \infty} 2ik (m_1(k, \xi, t) - 1).
\]
The rarefaction problem, results

**Theorem (Andreiev/E/Lange/Teschl ’16)**

For arbitrary small $\epsilon_j > 0$, $j = 1, 2, 3$, and for $\xi = \frac{x}{12t}$, the following asymptotics are valid as $t \to \infty$ uniformly with respect to $\xi$:

**A.** In the domain $(-6c^2 + \epsilon_1)t < x < -\epsilon_1 t$:

$$q(x, t) = -\frac{x + Q(\xi)}{6t}(1 + O(t^{-1/3})), \quad \text{as } t \to +\infty,$$

where

$$Q(\xi) = \frac{2}{\pi} \int_{-\sqrt{-2\xi}}^{\sqrt{-2\xi}} \left( \frac{d}{ds} \log R(s) - 4i \sum_{j=1}^{N} \frac{\kappa_j}{s^2 + \kappa_j^2} \right) \frac{ds}{\sqrt{s^2 + 2\xi}} \mp \frac{1}{2\sqrt{-2\xi}},$$

and $\pm$ corresponds to resonant/nonresonant cases.
B. In the domain $x < (-6c^2 - \epsilon_2) t$ in the nonresonant case:

$$q(x, t) = c^2 + \sqrt{\frac{4\nu y}{3t}} \sin(16ty^3 - \nu \log(192ty^3) + \delta)(1 + o(1)),$$

where $y = y(\xi) = \sqrt{\frac{c^2}{2} - \xi}$, $\nu = \nu(\xi) = -\frac{1}{2\pi} \log (1 - |R(y)|^2)$ and

$$\delta(\xi) = -\frac{3\pi}{4} + \arg(R(y) - 2T(y) + \Gamma(i\nu))$$

$$- \frac{1}{\pi} \int_{\mathbb{R}\setminus[-y,y]} \log \left( \frac{1 - |R(s)|^2}{1 - |R(y)|^2} \right) \frac{s ds}{s^2 - c^2 - (\frac{c^2}{2} + \xi)^{1/2}(c^2 - s^2)^{1/2}}.$$

Here $\Gamma$ is the Gamma-function.

C. In the domain $x > \epsilon_3 t$:

$$q(x, t) = -\sum_{j=1}^{N} \frac{2x_j^2}{\cosh^2 \left( x_j x - 4x_j^3 t - \frac{1}{2} \log \frac{\gamma_j(0)}{2x_j} - \sum_{i=j+1}^{N} \log \frac{x_i-x_j}{x_i+x_j} \right)} + O(e^{-\epsilon_3 t/2}).$$
The shock initial profile

\[ q_t(x, t) = 6q(x, t)q_x(x, t) - q_{xxx}(x, t), \quad (x, t) \in \mathbb{R} \times \mathbb{R}_+, \]

\[ q_0(x) \to \begin{cases} 
0, & x \to +\infty, \\
-c^2, & x \to -\infty,
\end{cases} \]

\[ \int_{0}^{+\infty} e^{C_0x}(|q_0(x)| + |q_0(-x) + c^2|)dx < \infty, \quad C_0 > c > 0. \]

The right scattering data, associated with the initial data:

\[ \{ R(k), \quad k \in \mathbb{R}; \quad |T(k)|, \quad k \in [0, ic]; \quad -\kappa_1^2, \ldots, -\kappa_p^2, \quad \gamma_1, \ldots, \gamma_p. \} \]
Statement of the RH problem

We consider the same regime: \( x \to \infty, \ t \to \infty, \ \frac{x}{12t} = \xi \), the nonresonant case only.

Introduce

\[
m(k) = (T(k, t)\phi_1(k, x, t)e^{ikx}, \ \phi(k, x, t)e^{-ikx}), \ k \in \mathbb{C}^+ \setminus (0, ic].
\]

Define it in \( \mathbb{C}^- \) by the symmetry condition

\[
m(k) = m(-k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

Asymptotic behavior as \( k \to +i\infty \)

\[
m(k) = (1, 1) - \frac{1}{2ik} \left( \int_x^{+\infty} q(y, t)dy \right) (-1, 1) + O \left( \frac{1}{k^2} \right).
\]
Theorem

Function $m(k)$ is the unique solution of the following RH problem: To find a vector function $m(k)$, meromorphic away from $\mathbb{R} \cup [-ic, ic]$, with simple poles at points $\pm ik_j$, which satisfies:

- The jump condition $m_+(k) = m_-(k)v(k)$, where

$$
v(k) = \begin{cases} 
\begin{pmatrix} 
1 - |R(k)|^2 & -R(k)e^{-2t\Phi(k)} \\
R(k)e^{2t\Phi(k)} & 1 
\end{pmatrix}, & k \in \mathbb{R}, \\
\begin{pmatrix} 
1 & 0 \\
\chi(k)e^{2t\Phi(k)} & 1 
\end{pmatrix}, & k \in [ic, 0), \\
\begin{pmatrix} 
1 & \chi(k)e^{-2t\Phi(k)} \\
0 & 1 
\end{pmatrix}, & k \in [0, -ic],
\end{cases}
$$

where $\chi(k) := -\sqrt{k^2 + c^2}/k |T(k)|^2$;
• the pole condition

\[ \text{Res}_{i\kappa_j} m(k) = \lim_{k \to i\kappa_j} m(k) \begin{pmatrix} 0 & 0 \\ i\gamma_j^2 e^{2t\Phi(i\kappa_j)} & 0 \end{pmatrix}, \]

\[ \text{Res}_{-i\kappa_j} m(k) = \lim_{k \to -i\kappa_j} m(k) \begin{pmatrix} 0 & -i\gamma_j^2 e^{2t\Phi(i\kappa_j)} \\ 0 & 0 \end{pmatrix}, \]

• the symmetry condition

\[ m(-k) = m(k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \]

• the normalizing condition

\[ \lim_{\kappa \to \infty} m(i\kappa) = (1 \quad 1). \]

Phase function: \[ \Phi(k) = \Phi(k, x, t) = 4ik^3 + ik \frac{x}{t} = 4ik^3 + 12ik\xi. \]
Region \(-c^2/2 < \xi < c^2/3, \xi = \frac{x}{12t}\)

Point \(a = a(\xi) \in (0, c)\) is given implicitly by the equation

\[
\int_a^0 \left( \xi + \frac{c^2 - a^2}{2} - s^2 \right) \sqrt{\frac{s^2 - a^2}{c^2 - s^2}} \, ds = 0.
\]

We conjugate with the "g-function":

\[
g(k) = 24i \int_{i\xi}^k \left( k^2 + \xi + \frac{c^2 - a^2}{2} \right) \sqrt{\frac{k^2 + a^2}{k^2 + c^2}} \, dk,
\]

\[
m^{(1)}(k) = m(k)e^{-t(\Phi(k) - g(k))} \sigma_3,
\]

and then make upper-lower and lower-upper factorizations.
Deformation of the contour and the model problem.
Model problem

\[ m_{\pm}^{\text{mod}}(k) = m_{-}^{\text{mod}}(k)v^{\text{mod}}(k), \]

\[ v^{\text{mod}}(k) = \begin{cases} 
\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, & k \in [ic, ia], \\
\begin{pmatrix} e^{-itB(\xi)+\Delta(\xi)} & 0 \\ 0 & e^{itB(\xi)-\Delta(\xi)} \end{pmatrix}, & k \in [ia, -ia], \\
\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, & k \in [-ia, -ic]
\end{cases} \]

Here \( B(\xi) = 2g(ia + 0) \), and \( \Delta(\xi) \) depend on the scattering data of the initial profile.
Let $\mathcal{M}$ be the Riemann surface of $\mathcal{R}(\lambda) = \sqrt{\lambda(\lambda + c^2)(\lambda + a^2)}$, with the sheets glued along intervals $[-c^2, -a^2] \cup [0, \infty)$, and let $p = (\lambda, \pm)$ denote a point on this surface.

Introduce the canonical basis of cycles.
Let $\mathcal{M}$ be the Riemann surface of $\mathcal{R}(\lambda) = \sqrt{\lambda(\lambda + c^2)(\lambda + a^2)}$, with the sheets glued along intervals $[-c^2, -a^2] \cup [0, \infty)$, and let $p = (\lambda, \pm)$ denote a point on this surface.

Introduce the canonical basis of cycles.

Let $d\omega$ be the holomorphic Abel differential, $\int_a d\omega = 2\pi i$;

Denote by $\mathcal{K}(\xi) = -\frac{\tau(\xi)}{2} + \pi i$ the Riemann constant, where $\tau(\xi) = \int_b d\omega$;

Let $A(p, \xi) = \int_\infty^p d\omega$ be the Abel map.
Let $\mathbb{M}$ — be the Riemann surface of $R(\lambda) = \sqrt{\lambda(\lambda + c^2)(\lambda + a^2)}$, with the sheets glued along intervals $[-c^2, -a^2] \cup [0, \infty)$, and let $p = (\lambda, \pm)$ denote a point on this surface.

Introduce the canonical basis of cycles.

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Let $A(p, \xi) = \int_\infty^p d\omega$ be the Abel map.

$$\Delta(\xi) = \frac{\int_b R^{-1}(\lambda) \left( \log |\chi(k)| + 2 \log \frac{k-i\kappa_j}{k+i\kappa_j} \right) d\lambda}{\int_a R^{-1}(\lambda) d\lambda}.$$
The following Jacobi inversion problem

\[ A(p, \xi) + K(\xi) = -\Delta(\xi) \]

has the unique solution \( p_0 = (\lambda_0, \pm) \), \( \lambda_0 \in [-a^2, 0] \).
The following Jacobi inversion problem

\[ A(p, \xi) + K(\xi) = -\Delta(\xi) \]

has the unique solution \( p_0 = (\lambda_0, \pm), \lambda_0 \in [-a^2, 0] \).

Let \( d\Omega_1 = \frac{i}{2\sqrt{\lambda}} \left(1 + O(\lambda^{-1})\right)\), \( d\Omega_3 = -6i\sqrt{\lambda} \left(1 + O(\lambda^{-2})\right)\) be the normalized Abel differentials of the second kind. Put

\[ V(\xi) = \int_b d\Omega_1, \quad W(\xi) = \int_b d\Omega_3, \quad \text{then} \]

\[ B(\xi) = V(\xi) + \xi W(\xi). \]
The following Jacobi inversion problem

\[ A(p, \xi) + K(\xi) = -\Delta(\xi) \]

has the unique solution \( p_0 = (\lambda_0, \pm) \), \( \lambda_0 \in [-a^2, 0] \).

Let \( d\Omega_1 = \frac{i}{2\sqrt{\lambda}} (1 + O(\lambda^{-1})) \), \( d\Omega_3 = -6i\sqrt{\lambda} (1 + O(\lambda^{-2})) \) be the normalized Abel differentials of the second kind. Put

\[ V(\xi) = \int_b d\Omega_1, \quad W(\xi) = \int_b d\Omega_3, \quad \text{then} \]

\[ B(\xi) = V(\xi) + \xi W(\xi). \]

Let

\[ \theta(v) := \theta(v, \xi) = \sum_{m \in \mathbb{Z}} \exp \left( \pi i m^2 \tau(\xi) + 2\pi i m v \right) \]

be the Jacobi theta-function.

Put \( h(\xi) = a^2(\xi) + c^2 + 2 \int_a^{\lambda} \mathcal{R}^{-1}(\lambda) d\lambda \left( \int_a^{\lambda} \mathcal{R}^{-1}(\lambda) d\lambda \right)^{-1}. \)
In the domain $-6c^2 t < x < 4c^2 t$ the solution has the following asymptotical behaviour as $t \to \infty$:

$$q(x, t) = -2 \frac{d^2}{dx^2} \log \theta \left( V(\xi)x - W(\xi)t - A(p_0, \xi) - K(\xi) \right) - h(\xi) + o(1).$$

In the domain $x > 4c^2 t$:

$$q(x, t) = -2 \sum_{j=1}^{N} \frac{\kappa_j^2}{\cosh^2(\kappa_j x - 4\kappa_j^3 t - p_j)} + O(t^{-1}),$$

where

$$p_j = \frac{1}{2} \log \left( \frac{\gamma_j^2}{2\kappa_j} \prod_{i=j+1}^{N} \left( \frac{\kappa_i - \kappa_j}{\kappa_i + \kappa_j} \right)^2 \right).$$
In the domain $x < -6c^2 t$

$$q(x, t) = -c^2 + P(x, t),$$

$$P(x, t) = \sqrt{\frac{4\nu(k_{1,0})k_{1,0}}{3t}} \sin(16tk_{1,0}^3 - \nu(k_{1,0}) \log(192tk_{1,0}^3) + \delta(k_{1,0})) + O(t^{-\alpha})$$

for any $1/2 < \alpha < 1$.

Here $k_{1,0} = \sqrt{\frac{c^2}{2} - \frac{x}{12t}},$

$$\nu(k_{1,0}) = -\frac{1}{2\pi} \log(1 - |R_1(k_{1,0})|^2),$$

$$\delta(k_{1,0}) = \frac{\pi}{4} - \arg(R_1(k_{1,0})) + \arg(\Gamma(i\nu(k_{1,0}))) - \frac{1}{\pi} \int_{\mathbb{R}\setminus[-k_{1,0}, k_{1,0}]} \log \left( \frac{1 - |R_1(\zeta)|^2}{1 - |R_1(k_{1,0})|^2} \right) \frac{1}{\zeta - k_{1,0}} d\zeta.$$
We consider the Cauchy problem for the Toda lattice equation

\[ \begin{align*}
\dot{b}(n, t) &= 2(a^2(n, t) - a^2(n - 1, t)), \\
\dot{a}(n, t) &= a(n, t)(b(n + 1, t) - b(n, t)),
\end{align*} \]

with the steplike initial data

\[ \begin{align*}
a(n, 0) &\to a_+, \quad b(n, 0) \to b_+, \quad \text{as } n \to +\infty, \\
a(n, 0) &\to a_-, \quad b(n, 0) \to b_-, \quad \text{as } n \to -\infty.
\end{align*} \]

The Toda lattice equation is equivalent to the Lax equation \( \dot{\mathcal{H}} = [\mathcal{H}, \mathcal{A}] \):

\[ \begin{align*}
(\mathcal{H}(t)y)(n) &:= a(n - 1, t)y(n - 1) + b(n, t)y(n) + a(n, t)y(n + 1), \\
(\mathcal{A}(t)y) &:= -a(n - 1, t)y(n - 1) + a(n, t)y(n + 1).
\end{align*} \]
Background spectra

The Jacobi equation:

\[ a(n - 1, t)y(n - 1) + b(n, t)y(n) + a(n, t)y(n + 1) = \lambda y(n). \]

By shifting and scaling of the spectral parameter the initial data of a general location are:

\[ a(n, 0) \rightarrow \frac{1}{2}, \quad b(n, 0) \rightarrow 0, \quad \text{as} \quad n \rightarrow +\infty, \]
\[ a(n, 0) \rightarrow a, \quad b(n, 0) \rightarrow b, \quad \text{as} \quad n \rightarrow -\infty. \]

Background operators

\[(H_1y)(n) := ay(n - 1) + by(n) + ay(n + 1),\]
\[(Hy)(n) := \frac{1}{2}y(n - 1) + \frac{1}{2}y(n + 1), \quad n \in \mathbb{Z},\]

define the continuous spectrum of \( \mathcal{H}(t) \).
The Toda shock problem

Let the constants $b, a \in \mathbb{R}$ satisfy the condition $b < -1$, $0 < 2a < -b - 1$. Then the background spectra $[b - 2a, b + 2a]$ and $[-1, 1]$ are not overlapping and the left background spectrum is located to the left of the right background spectrum, that corresponds to the Toda shock problem.

- Direct/inverse scattering - Oba ’91, E/ Michor/ Teschl ’08
- The Cauchy problem solution - E/Michor/Teschl ’09
- Long time asymptotics (for $a=1/2$, in the middle interval) - Venakides/ Deift/ Oba ’91
- Five regions, divided by rays $\frac{n}{t} = \xi_{cr}$: $\xi_{cr,1} < \xi'_{cr,1} < \xi'_{cr,2} < \xi_{cr,2}$.
Non-overlapping background: $\sigma(H) = [-1, 1]$, $\sigma(H_1) = [1.2, 2.8]$, $a = 0.4$, $b = 2$. 

\begin{align*}
\text{a}(n,200) & \quad \text{b}(n,200)
\end{align*}
Toda rarefaction problem, RH problem approach

\[ a(n, 0) \to a, \quad b(n, 0) \to b, \quad \text{as} \ n \to -\infty, \]
\[ a(n, 0) \to \frac{1}{2}, \quad b(n, 0) \to 0, \quad \text{as} \ n \to +\infty, \]

where \( a > 0, \ b \in \mathbb{R} \) satisfy the condition \( 1 < b - 2a \).

Results (E/Michor/Teschl):

- In the region \( n > t \), the solution \{a(n, t), b(n, t)\} is asymptotically close to the coefficients of the right background Jacobi operator \{\frac{1}{2}, 0\}, plus a sum of solitons corresponding to the eigenvalues \( \lambda_j < -1 \).
- In the region \( n < -2at \), the solution is close to the left background constants \{a, b\}, plus a sum of solitons corresponding to the eigenvalues \( \lambda_j > b + 2a \).
In the region $-2at < n < 0$, as $t \to \infty$ we have

\[
a(n, t) = -\frac{n}{2t} + O\left(\frac{1}{t}\right), \quad b(n, t) = b - 2a - \frac{n}{t} + O\left(\frac{1}{t}\right).
\]

In the region $0 < n < t$, as $t \to \infty$ we have

\[
a(n, t) = \frac{n}{2t} + O\left(\frac{1}{t}\right), \quad b(n, t) = 1 - \frac{n}{t} + O\left(\frac{1}{t}\right).
\]

\[
O\left(\frac{1}{t}\right) = -\frac{n}{8t^2} \left(\frac{\sqrt{1 - \frac{n}{t}} + 1 + \frac{2n}{t}}{\frac{n}{t} \sqrt{1 - \frac{n}{t}} (1 + \sqrt{1 - \frac{n}{t}})}\right)
\]

\[
\frac{1}{\sqrt{2\pi}} \int_{\theta_0}^{2\pi - \theta_0} \frac{2u'(\theta) \sin \frac{\theta}{2} - u(\theta) \cos \frac{\theta}{2}}{\sqrt{1 - \frac{2n}{t} - \cos \theta \sin^2 \frac{\theta}{2}}} d\theta \left(1 + o(1)\right)
\]

\[
\theta_0 = \arccos\left(1 - \frac{n}{2t}\right), \quad u(\theta) = R(\cos \theta) \left(\prod_{z_k \in \{0,1\}} |z_k|\frac{e^{i\theta} - z_k^{-1}}{e^{i\theta} - z_k}\right)^{-2}.
\]
Embedding backgrounds: $\sigma(H) = [-1, 1], \sigma(H_1) = [-0.1, 0.7]$. 
Overlapping backgrounds: $\sigma(H) = [-1, 1]$, $\sigma(H_1) = [-4, 0]$; $a = 1$, $b = -2$. 
Overlapping backgrounds: $\sigma(H) = [-1, 1]$, $\sigma(H_1) = [-0.9, 1.5]$; $a = 0.6$, $b = 0.3$. 
Thank you for your attention!