Logvinenko-Sereda Theorems for periodic functions

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Content

1. Introduction

Logvinenko-Sereda Theorems

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Let $S \subset \mathbb{R}$. S is (γ, a)-*thick* if there exist $\gamma > 0$ and $a > 0$ such that for all I intervals of $\mathbb{R}$ of length a we have

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Theorem 1 (Logvinenko-Sereda '78)

Let $J \subset \mathbb{R}$ of length $b > 0$. If $f \in L^p(\mathbb{R})$, $p \in [1, \infty]$, with $\operatorname{supp} \widehat{f} \subset J$ and if S is (γ, a)-thick, then

$$\|f\|_{L^p(S)} \geq \exp\Big(-C\frac{ab+1}{\gamma}\Big)\|f\|_{L^p(\mathbb{R})}.$$

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- The position of J is irrelevant.

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Theorem 2 (Kovrijkine, '01)

Under the same assumptions of Thm 1, we have

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- Extended support assumption

Theorem 3 (Kovrijkine, '01)

Let $n \in \mathbb{N}$, $f \in L^p(\mathbb{R})$ *with* $p \in [1, \infty]$ *and assume that* $\operatorname{supp} \hat{f} \subset J_1 \cup \ldots \cup J_n$, *where each* J_k *is an interval of length* b. *Let* S *be a* (γ, a)*-thick set. Then,*

$$\|f\|_{L^p(S)} \geq \left(\frac{\gamma}{C}\right)^{ab\left(\frac{C}{\gamma}\right)^n + n - \frac{p-1}{p}} \|f\|_{L^p(\mathbb{R})}. \tag{1}$$

Questions

Question 1: Let S be (γ, a)-thick. Let $E_0 \in \mathbb{R}$ and $f \in \operatorname{Ran}(\chi_{(E_0-1,E_0]}(\Delta_{\mathbb{T}_L}))$. Does the estimate

$$\int_S |f|^2 \geqslant C(\gamma, a) \int_{\mathbb{T}_L} |f|^2, \tag{2}$$

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Answer: Yes!

2. Results

Results

Theorem 4

Let $\mathbb{T}_L = [0, 2\pi L]$, $f \in L^p(\mathbb{T}_L)$ *with* $p \in [1, \infty]$ *such that* $\operatorname{supp} \hat{f} \subset J$, *where* $J \subset \mathbb{R}$ *is an interval of length* b. *Let* $S \subset \mathbb{R}$ *be a* (γ, a)*-thick set with* $0 < a \leq 2\pi L$. *Then,*

$$\|f\|_{L^p(\mathbb{T}_L)} \leq \left(\frac{c_1}{\gamma}\right)^{c_2 ab + \frac{3}{p}} \|f\|_{L^p(S \cap \mathbb{T}_L)}, \tag{3}$$

where c_1, c_2 *are universal constants.*

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- $\widehat{f} : \frac{1}{L}\mathbb{Z} \longrightarrow \mathbb{R}$, $\widehat{f}\left(\frac{k}{L}\right) = \frac{1}{2\pi L} \int_{\mathbb{T}_L} f(x) e^{-i\frac{k}{L}x}\, dx$, $k \in \mathbb{Z}$.

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- The bigger L, the more active Fourier coefficients are in J, but this does not affect the estimate
- The position of J is irrelevant.

Results

Theorem 5

Let $f \in L^p(\mathbb{T}_L)$ with $p \in [1, \infty]$. Assume that $\operatorname{supp} \widehat{f} \subset \bigcup_{l=1}^{n} J_l$, *where* $J_l \subset \mathbb{R}$ *are intervals of length* b. *Let* S *be a* (γ, a)*-thick set with* $0 < a \leq 2\pi L$. *Then,*

$$\|f\|_{L^p(\mathbb{T}_L)} \leq \left(\frac{\tilde{c}_1}{\gamma}\right)^{\left(\frac{\tilde{c}_2}{\gamma}\right)^n ab + n - \frac{(p-1)}{p}} \|f\|_{L^p(S \cap \mathbb{T}_L)}, \tag{4}$$

where $\tilde{c}_1$ *and* $\tilde{c}_2$ *are universal constants.*

3. Application to PDE

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Assume $J \subset \mathbb{R}$ such that $|J| = b$ and consider

$$\Delta_{\mathbb{T}_L} u_k = -\lambda_k^2 u_k \quad \text{with periodic boundary conditions.} \tag{5}$$

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Corollary 6

Let f be a linear combination of eigenfunctions of (5), i.e.,

$$f(x) = \sum_{\frac{k}{L} \in \frac{1}{L}\mathbb{Z} \cap J} c_{\frac{k}{L}} e^{i\frac{k}{L}x}$$

and let S be (γ, a)-thick. Then,

$$\|f\|_{L^2(S \cap \mathbb{T}_L)} \geq \left(\frac{\gamma}{c_1}\right)^{c_2 ab + \frac{3}{2}} \|f\|_{L^2(\mathbb{T}_L)}. \tag{6}$$

4. Application to Control Theory

A warm-up result - the uncontrolled heat equation

We consider the uncontrolled heat equation with periodic boundary conditions

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } (0,T) \times \mathbb{T}_L \\ \frac{\partial^m u}{\partial x^m}(t,x) = \frac{\partial^m u}{\partial x^m}(t,x) \quad \forall\, m \geq 0 & \text{in } (0,T) \times \partial\mathbb{T}_L \\ u(0,x) = u_0 \in V & \text{in } \mathbb{T}_L, \end{cases} \tag{7}$$

where $V = \{f \in L^2(\mathbb{T}_L) \mid \operatorname{supp} \widehat{f} \subset \bigcup_{l=1}^{n} J_l\}$ with J_l's intervals of length $b > 0$.

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Proposition 7

The solution of the above system satisfies the observability inequality

$$\|u(T,\cdot)\|^2_{L^2(\mathbb{T}_L)} \leq \frac{1}{T}\left(\frac{\tilde{c}_1}{\gamma}\right)^{2\left(\frac{\tilde{c}_2}{\gamma}\right)^n ab + 2n - 1} \|u\|^2_{L^2((0,T)\times\omega)},$$

where $\omega = S \cap \mathbb{T}_L$ and S is (γ, a)-thick.

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Following [Le Roussea-Lebeau '12], we observe that

$$u(0,x) = \sum_{\frac{k}{L} \in \bigcup_{l=1}^{n} J_l} \beta_{k/L} u_{k/L}(x)$$

where $u_{k/L}$ are the eigenfunctions of the periodic Laplacian on $\mathbb{T}_L$ arising from the active Fourier modes,

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Consequently, we have

$$u(t,x) = \sum_{\frac{k}{L} \in \bigcup_{l=1}^{n} J_l} e^{-tE_k} \beta_{k/L} u_{k/L}(x) = \sum_{\frac{k}{L} \in \bigcup_{l=1}^{n} J_l} \beta_{k/L}(t) u_{k/L}(x),$$

where E_k's denote the eigenvalues arising from the active Fourier modes.

Then,

$$T\|u(T,\cdot)\|^2_{L^2(\mathbb{T}_L)} \leq \int_0^T \int_{\mathbb{T}_L} |u(t,x)|^2 \, \mathrm{d}\,x \, \mathrm{d}\,t$$

Then,

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Then,

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\begin{aligned}
T\|u(T,\cdot)\|^2_{L^2(\mathbb{T}_L)} &\le \int_0^T \int_{\mathbb{T}_L} |u(t,x)|^2 \,\mathrm{d}\,x \,\mathrm{d}\,t \\
&= \int_0^T \int_{\mathbb{T}_L} \Big| \sum_{\frac{k}{L}\in\bigcup_{l=1}^n J_l} \beta_{k/L}(t) u_{k/L}(x) \Big|^2 \,\mathrm{d}\,x \,\mathrm{d}\,t \\
&\le \left(\frac{\tilde{c}_1}{\gamma}\right)^{2\left(\frac{\tilde{c}_2}{\gamma}\right)^n ab+2n-1} \int_0^T \int_{S\cap\mathbb{T}_L} \Big| \sum_{\frac{k}{L}\in\bigcup_{l=1}^n J_l} \beta_{k/L}(t) u_{k/L}(x) \Big|^2 \,\mathrm{d}\,x \,\mathrm{d}\,t
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\end{aligned}
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and so

$$
\|u(T,\cdot)\|^2_{L^2(\mathbb{T}_L)} \leq \frac{1}{T}\left(\frac{\tilde{c}_1}{\gamma}\right)^{2\left(\frac{\tilde{c}_2}{\gamma}\right)^n ab+2n-1} \|u\|^2_{L^2((0,T)\times\omega)}.
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The controlled heat equation

Let S be (γ, a)-thick and consider the following controlled heat equation.

$$\begin{cases} \partial_t u - \Delta u = v\chi_\omega & \text{in } (0,T)\times\mathbb{T}_L, \\ & \omega = (0,T)\times(\mathbb{T}_L \cap S) \\ \frac{\partial^m u}{\partial x^m}(t,x) = \frac{\partial^m u}{\partial x^m}(t,x) \quad \forall\, m \geq 0 & \text{in } (0,T)\times\partial\mathbb{T}_L \\ u(x,0) = u_0(x) \in L^2(\mathbb{T}_L) \end{cases} \tag{8}$$

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Null-controllability: driving the solution u to zero at time $T > 0$, yet only acting in the sub-domain ω. (For null-controllability results on general non-empty set ω, see [Lebeau-Robbiano '95] and [Fursikov-Immanuvilov '96].

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Control-cost: $\inf\{\|v\|_{L^2((0,T)\times\omega)} \mid u \text{ sol. of (8) and } u(\cdot,T)=0\}$.

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Goal: estimate the control cost with respect to the geometric properties of the subset ω.

Controlled heat equation - Partial result

Lemma 8

Consider

$$\begin{cases} \partial_t u - \Delta u = \Pi_F(v\chi_\omega) & \text{in } (0,T)\times\mathbb{T}_L, \omega = (0,T)\times(\mathbb{T}_L \cap S) \\ u(x,0) = u_0(x) \in F \end{cases}$$

with periodic boundary condition on $\partial\mathbb{T}_L \times (0,T)$*, and where* $F = \text{span}\{\phi_{k/L} \text{ eigenfunction of } \Delta_{\mathbb{T}_L} \mid (k/L)^2 \le b^2\}$*. Then, there exists a* v *that drives the solution to zero and such that*

$$\|v\|_{L^2((0,T)\times\omega)} \le \frac{1}{\sqrt{T}}\left(\frac{c_1}{\gamma}\right)^{c_2 ab+\frac{3}{2}} \|u_0\|_{L^2(\mathbb{T}_L)}. \tag{9}$$

Π_F is the orthogonal projection on F.

Proof:

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- By duality between null-controllability and observability, we consider

$$\begin{cases} -\partial_t q - \Delta q = 0 & \text{in } (0,T) \times \mathbb{T}_L \\ q(T) = q_f \in F \end{cases} \tag{10}$$

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  with periodic boundary conditions on $(0,T)\times\partial\mathbb{T}_L$.
- Use same method of the proof of Prop. 7 to obtain

$$T\|q(0)\|^2_{L^2(\mathbb{T}_L)} \leq \left(\frac{c_1}{\gamma}\right)^{c_2ab+3} \|q\|_{L^2((0,T)\times\omega)}.$$

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- This already yields

$$\|v\|_{L^2((0,T)\times\omega)} \leq \frac{1}{\sqrt{T}} \left(\frac{c_1}{\gamma}\right)^{c_2 ab+\frac{3}{2}} \|u_0\|_{L^2(\mathbb{T}_L)}.$$

Controlled heat equation - Full result

Proposition 9

Consider the following system

$$\begin{cases} \partial_t u - \Delta u = v\chi_\omega & \text{in } (0,T)\times\mathbb{T}_L, \\ & \omega = (0,T)\times(\mathbb{T}_L\cap S) \\ \frac{\partial^m u}{\partial x^m}(t,x) = \frac{\partial^m u}{\partial x^m}(t,x) \quad \forall\, m\geq 0 & \text{in } (0,T)\times\partial\mathbb{T}_L \\ u(x,0) = u_0(x)\in L^2(\mathbb{T}_L) \end{cases}$$

Then, there exists a v that drives the solution to zero and such that

$$\|v\|_{L^2((0,T)\times\omega)} \leq C(T,a,\gamma)\|u_0\|_{L^2(\mathbb{T}_L)} \tag{11}$$

Controlled heat equation - Full result

Proposition 9

Consider the following system

$$\begin{cases} \partial_t u - \Delta u = v\chi_\omega & \text{in } (0,T)\times\mathbb{T}_L, \\ & \omega = (0,T)\times(\mathbb{T}_L\cap S) \\ \frac{\partial^m u}{\partial x^m}(t,x) = \frac{\partial^m u}{\partial x^m}(t,x) \quad \forall\, m\geq 0 & \text{in } (0,T)\times\partial\mathbb{T}_L \\ u(x,0) = u_0(x)\in L^2(\mathbb{T}_L) \end{cases}$$

Then, there exists a v that drives the solution to zero and such that

$$\|v\|_{L^2((0,T)\times\omega)} \leq C(T,a,\gamma)\|u_0\|_{L^2(\mathbb{T}_L)} \tag{11}$$

- $C(T,a,\gamma)\sim e^{-1/T}$

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- $C(T,a,\gamma)$ is still polynomial in γ

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  On $(a_j, aj + T_j]$ we solve a controlled heat equation with a control v that satisfies Lemma 8 with initial data in F_j.
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  On both part, we estimate the L^2-norm of the solution.
- 'Glue together' the two estimates to infer that the solution goes to zero at time T.
- Write $\|v\|^2_{L^2((0,T)\times\mathbb{T}_L)} = \sum_{j\geq 0} \|v\|^2_{L^2((a_j,a_j+T_j]\times\mathbb{T}_L)}$ and use Lemma 8 on each interval together with estimates of the solution to get the final estimate.

5. Sketch of proof of Them 4

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  Shift J by $c \in \frac{1}{L}\mathbb{Z} \cap J$ so that $J - c \subset [-b, b]$.
- Cover $\mathbb{T}_L = \bigcup_{j=1}^{N} I_j$, $|I_j| = 1$.
- Let $A > 1$. I_j is bad if there exists $\alpha \geq 1$ such that

$$\|f^{(\alpha)}\|^p_{L^p(I_j)} \geq A^\alpha (Cb)^\alpha \|f\|^p_{L^p(I_j)},$$

  and I_j is good if for all $\alpha \geq 1$ we have

$$\|f^{(\alpha)}\|^p_{L^p(I_j)} \leq A^\alpha (Cb)^\alpha \|f\|^p_{L^p(I_j)},$$

  Then, we can prove that $\|f\|^p_{L^p(\cup_{I \text{ bad}} I_j)} \leq \frac{1}{2}\|f\|^p_{L^p(\mathbb{T}_L)}$ so we can discard the bad boxes.

- It is all about proving

$$\|f\|^p_{L^p(I_j\cap S)} \leq \left(\frac{\gamma}{c_1}\right)^{pc_2b+3} \|f\|^p_{L^p(I_j)},$$

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  - $\exists\ B > 1$ such that if I_j is good there exists $x_0 \in I_j$ such that

$$|f^{(\alpha)}(x_0)| \leq 2B^{\alpha p}(CB)^{\alpha p} \int_{I_J} |f|^p \qquad \forall\ \alpha > 0.$$

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  - Assume $I_j = [-1/2, 1/2]$ by shifting f by an appropriate n. Then, if $z \in D(x, R+1/2)$ for $x \in I_j$, using Taylor's series we have

$$|f(z)| \le 2^{1/p} \exp(Cb(R+1/2))\|f\|^p_{L^p(I_j)} \tag{12}$$

- Using a technical lemma for analytic functions, we conclude

$$\int_{I_j \cap S} |f|^p \geq \left(\frac{|S \cap I_j|}{C} \right)^{2p \log M / \log 2 + 1} \int_{I_j} |f|^p$$
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where $M \leq \max_{|z| \leq 4+1/2} |f(z)| \leq 2^{1/p} \exp(5CB)$.

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Lemma 10

Let $p \in [1, \infty)$ *and let* ϕ *be an analytic function on* $D(0,5) := \{z \in \mathbb{C} \mid |z| < 5\}$*. Let* $I \subset \mathbb{R}$ *be an interval of unit length such that* $0 \in I$ *and let* $A \subset I$ *be a measurable set of non-zero measure, i.e.,* $|A| > 0$*. Set* $M = \max_{|z| \leq 4} |\phi(z)|$ *and assume that* $|\phi(0)| \geq 1$*, then*

$$\int_I |\phi|^p \leq \left(\frac{C}{|A|} \right)^{2p \frac{\log M}{\log 2} + 1} \int_A |\phi|^p.$$

THANK YOU FOR YOUR ATTENTION