Regularized integrals of motion for the Korteweg-de Vries equation in a class of non-decreasing functions

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In the work [1] the Cauchy problem for the KdV equation was solved

$$u_t(x,t) = 6u(x,t)u_x(x,t) - u_{xxx}(x,t),$$

with the initial condition

$$u|_{t=0} = u_0(x),$$

where $u_0(x)$ is real, the fast decreasing initial function

$$u_0(x) \to 0, \quad x \to \pm\infty.$$

Two important observations were made after this paper.

[1] *Gardner C.S., Greene J.M., Kruskal M.D., Miura R.M.* Method for solving the Korteweg-de Vries equation // Phys. Rev. Lett. 19 (1967), 1095-1097.

1. The integrals of motion

The KdV equation with smooth initial data has an infinite set of first integrals

$$I_n[u] = \int_{-\infty}^{\infty} P_n(u, u_x, ..., u_x^{(n-2)})dx, \quad \frac{dI_n[u]}{dt} = 0,$$

where P_n is a polynomial with respect to all its variables, that is with u and the space derivatives of u. The first three of these polynomials have the form

$$P_1(u) = u, \quad P_2(u) = u^2, \quad P_3(u, u_x) = u^3 + \frac{1}{2}u_x^2.$$

- *[2] Miura R.M., Gardner C.S., Kruskal M.D.* Korteweg-de Vries equation and generalizations, II. Existence of conservation laws and constans of motion. // J. Math. Phys. **9**, No.8 (1968), 1204-1209.
- *[3] Kruskal M.D., Miura R.M., Gardner C.S., Zabusky N.J.* Korteweg-de Vries equation and generalizations, V. Uniqueness and nonexistence of polynomial conservation laws // J. Math. Phys. **11**, No.3 (1970), 952-960.
- *[4] Lax P.D.* Integrals of nonlinear equations and solitary waves, // Comm. Pure Appl. Math. **21**, No.2 (1968), 467-490.

2. The KdV equation is the completely integrable Hamiltonian system

It was shown that the Hamiltonian $H[u]$ is one of the integrals of motion, i.e. the KdV equation can be represented in the form

$$\frac{du}{dt} = \frac{d}{dx}\frac{\delta H[u]}{\delta u},$$

where $\frac{\delta H[u]}{\delta u(x)}$ is the Frechet derivative. The third integral of motion is the Hamiltonian

$$H[u] = I_3[u] = \int_{-\infty}^{\infty} \left(u^3 + \frac{1}{2} u_x^2 \right) dx.$$

Completely integrable system is understood in the Liouville sense

$$\{I_i, I_j\} = 0, \quad i \neq j,$$

where $\{\cdot, \cdot\}$ is the Poisson brackets.

[5] *Zakharov V.E., Faddeev, L.D.* Korteweg-de Vries equation: A completely integrable Hamiltonian system. Functional analysis and its applications // 1971, **5**: No.4: 18-27 (in Russian).

Elements of the scattering theory

Consider the spectral Schrödinger equation

$$-\frac{d^2}{dx^2}y(x)+u_0(x)y(x)=k^2y(x), \quad -\infty<x<\infty.$$

We suppose that the potential belongs to the Schwartz class $u_0\in\Sigma_0$:

$$\Sigma_0=\left\{u: \quad \int_{-\infty}^{\infty}(1+|x|^m)u^{(j)}(x)dx<\infty, \quad m,j\in\mathbb{N}\right\}.$$

The Schödinger operator has a continuous spectrum of multiplicity two on $\mathbb{R}_+$ and a finite number of negative eigenvalues $-\varkappa_l^2, l=\overline{1,N}$. This equation has a solution $y(x,k)$ which uniquely defined by the asymptotical behavior

$$y(x,k)=\begin{cases} e^{\mathrm{i}kx}+R(k)e^{-\mathrm{i}kx}+o(1), & x\to-\infty,\\ T(k)e^{\mathrm{i}kx}+o(1), & x\to+\infty.\end{cases}$$

where $R(k)$, $T(k)$ are the reflection and transmission coefficients.

Elements of the scattering theory

- Let $y_l(x),\ l=\overline{1,N}$ be eigenfunctions of discrete spectrum, normalized by the condition $y_l(x)=e^{\varkappa_l x}(1+o(1)),\ x\to-\infty$;
- Let $\gamma_l, l=\overline{1,N}$ be the corresponding weights $\gamma_l^{-2}=\int_{-\infty}^{\infty} y_l^2(x)dx$.

The set $\mathcal{S}=\{R(k),\ k\in\mathbb{R};\ -\varkappa_l^2;\ \gamma_l>0,\ l=\overline{1,N}\}$ is called the scattering data.

$$\mathcal{S}(0)=\{R(k,0),k\in\mathbb{R};\varkappa_l;\gamma_l(0)\}\ \xrightarrow[\text{evolution}]{\text{the time}}\ \mathcal{S}(t)=\{R(k,t),k\in\mathbb{R};\varkappa_l;\gamma_l(t)\}$$

the direct problem ↑ $u(x,0)$; the inverse problem ↓ $u(x,t)$

$$u(x,0)\ \xrightarrow{\text{the KdV equation}}\ u(x,t)$$

The time evolution

- the reflection coefficient: $R(k,t)=R(k,0)e^{8ik^3t}$,
- the discrete spectrum: $\varkappa_l(t)=\varkappa_l(0),\ l=\overline{1,N}$,
- the weights: $\gamma_l(t)=\gamma_l(0)e^{8\varkappa_l^3 t},\ l=\overline{1,N}$.

The symplectic form

The symplectic form Ω on $u(x) \in \Sigma_0$ is

$$\Omega(\delta_1 u, \delta_2 u) = \int_{-\infty}^{\infty} dx \int_{x}^{\infty} dy \Big(\delta_1 u(x) \cdot \delta_2 u(y) - \delta_1 u(y) \cdot \delta_2 u(x)\Big),$$

where $\delta u(x)$ is a variation of $u(x)$ ($\delta u(x) \in \Sigma_0$, Σ_0 is the Schwartz class).
The Hamiltonian on manifold Σ_0 is defined by equality

$$H[u] = \int_{-\infty}^{+\infty} (u^3(x) + \frac{1}{2} u_x^2) dx,$$

and the KdV equation has the form

$$u_t = \frac{d}{dx} \frac{\delta H[u]}{\delta u}.$$

The map $u \to \mathcal{S}$ is a transformation from potential $u(x)$ to the scattering data. In [5] the symplectic form Ω and the Hamiltonian $H[u]$ were represented in terms of the scattering data.

[5] *Zakharov V.E., Faddeev, L.D.* Korteweg-de Vries equation: A completely integrable Hamiltonian system. Functional analysis and its applications // 1971, **5**: No.4: 18-27 (in Russian).

From the symplectic form to the scattering data

$$\delta_1 u(x) \to \delta_1 \mathcal{S}, \quad \delta_2 u(x) \to \delta_2 \mathcal{S},$$
$$\Omega(\delta_1 u, \delta_2 u) = \Omega_{\mathcal{S}}(\delta_1 \mathcal{S}, \delta_2 \mathcal{S}),$$

where

$$\Omega_{\mathcal{S}}(\delta_1 \mathcal{S}, \delta_2 \mathcal{S}) := \int_{-\infty}^{+\infty} \Big(\delta_1 P(k)\delta_2 Q(k,t) - \delta_1 Q(k,t)\delta_2 P(k)\Big)\, dk + \sum_{l=1}^{N} (\delta_1 p_l \delta_2 q_l - \delta_1 q_l \delta_2 p_l),$$

$$P(k) := -\frac{k}{\pi} \log(1 - |R(k,t)|^2), \quad Q(k,t) := \arg T(k,t) - \arg R(k,t),$$

$$p_l := \varkappa_l^2, \quad q_l(t) := 2 \log\left(\mathrm{i}\gamma_l \frac{d}{dk} T^{-1}(k,t)|_{k=\mathrm{i}\varkappa_l} \right), \quad l = \overline{1,N}.$$

Zaharov V.E., Faddeev L.D.'71

The variables $P(k)$, p_l, $Q(k,t)$, $q_l(t)$ are the canonical variables of "action-angle" type. The Hamiltonian can be expressed in terms of the "action" variables $P(k,t)$, p_l:

$$H[u] = \frac{1}{2\pi} \int_{-\infty}^{\infty} k^3 P(k) dk - \frac{32}{5} \sum_{l=1}^{N} p_l^{3/2}.$$

In the canonical variables the KdV equation looks like

$$\frac{d}{dt} P(k) = 0, \quad \frac{d}{dt} p_l = 0, \quad \frac{d}{dt} Q(k,t) = 8k^3, \quad \frac{d}{dt} q_l(t) = -8\varkappa_l^3.$$

The KdV equation in a class of non-decreasing functions

Consider the Cauchy problem for the KdV equation

$$u_t(x,t) = 6u(x,t)u_x(x,t) - u_{xxx}(x,t), \quad u|_{t=0} = u_0(x),$$

with real infinitely differentiable the initial function $u_0(x) \in \Sigma, \Sigma$ is manifold of the Schwartz type

$$\Sigma = \Big\{ u : \quad \int_{-\infty}^{0} (1+x^m)|u^{(j)}(x) - v^{(j)}(x)|dx < \infty,$$
$$\int_{0}^{\infty} (1+x^m)|u^{(j)}(x)|dx < \infty, \quad j,m = 0,1,... \Big\},$$

where $v(x)$ is real periodic finite-gap potential for the Hill equation

$$-y'' + v(x)y = k^2 y, \quad -\infty < x < \infty,$$

with period $1 : v(x+1) = v(x)$.

Problem 1

To construct the set of the integrals of motion for such type equations.

Regularized integrals of motion

Result 1 (K.Andreiev'15)

There is an infinite set of the regularized integrals of motion that depend on time

$$I_j[u,t] = \int_{-\infty}^{0} Q_j[u]d\xi + \int_0^{\infty} \sigma_j[u]d\xi + \int_0^t P_j[v(0,\tau)]d\tau,$$

$$\frac{dI_j}{dt} = \frac{\partial I_j}{\partial t} + \frac{\delta I_j}{\delta u}\frac{\partial u}{\partial t} = 0, \quad j = 1, 2, \dots$$

where $v(x,t)$ is the periodic solution of the KdV equation with initial function $v(x,0) = v_0(x)$; $v(x+1,t) = v(x,t)$, and the functions $Q_j(\xi,t), \sigma_j(\xi,t), P_j(t)$ are polynomial with respect to u, v and derivatives of u, v by the space variable

$$Q_j[u] = Q_j(u(\xi,t), u_\xi(\xi,t), \dots),$$

$$\sigma_j[u] = \sigma_j(u(\xi,t), u_\xi(\xi,t), \dots),$$

$$P_j[v(0,\tau)] = P_j(v(0,\tau), v_\xi(0,\tau), \dots).$$

The integrals of motion in the periodic case

$$I_1[u,t] = \int_{-\infty}^{0} (u(\xi,t) - v(\xi,t))\, d\xi + \int_0^{\infty} u(\xi,t) d\xi + \int_0^t \left(3v^2(0,\tau) - v_{\xi\xi}(0,\tau)\right) d\tau;$$

$$I_2[u] = v(0,0);$$

$$I_3[u,t] = \int_{-\infty}^{0} \left(-u^2(\xi,t) + v^2(\xi,t) + u_{\xi\xi}(\xi,t) - v_{\xi\xi}(\xi,t)\right) d\xi + \int_0^{\infty} \left(-u^2(\xi,t) + u_{\xi\xi}(\xi,t)\right) d\xi$$

$$+ \int_0^t \Big(- 4v^3(0,\tau) + 8v(0,\tau)v_{\xi\xi}(0,\tau) + 5v_\xi^2(0,\tau) - v^{(4)}(0,\tau)\Big) d\tau;$$

$$I_4[u] = -2v^2(0,0) + v_{xx}(0,0);$$

$$I_5[u,t] = \int_{-\infty}^{0} \left(2u^3(\xi,t) - 2v^3(\xi,t) + u_\xi^2(\xi,t) - v_\xi^2(\xi,t) + u^{(4)}(\xi,t) - v^{(4)}(\xi,t)\right) d\xi$$

$$+ \int_0^{\infty} \left(2u^3(\xi,t) + u_\xi^2(\xi,t) + u^{(4)}(\xi,t)\right) d\xi$$

$$+ \int_0^t \Big(9v^4(0,\tau) - 42v^2(0,\tau)v_{\xi\xi}(0,\tau) - 60v(0,\tau)v_\xi^2(0,\tau) + 12v(0,\tau)v^{(4)}(0,\tau)$$

$$+ 28v_\xi(0,\tau)v^{(3)}(0,\tau) + 19v_{\xi\xi}^2(0,\tau) - v^{(6)}(0,\tau)\Big) d\tau;$$

$$I_6[u] = \frac{16}{3} v^3(0,0) - 8vv_{xx}(0,0) - 5v_x^2(0,0) + v^{(4)}(0,0).$$

and the KdV equation can be represented as $\frac{du}{dt} = \frac{d}{dx}\frac{\delta I_5[u,t]}{\delta u}$.

The integrals of motion in the steplike case

Consider the case when the solution of the Cauchy problem tends to a constant background at $x \to -\infty$, that is $v(x,t) = const = c^2,\ c \in \mathbb{R}\backslash\{0\}$.

$$\begin{aligned}
I_1[u,t] =& \int_{-\infty}^{0} \left(u(\xi,t) - c^2\right) d\xi + \int_0^{\infty} u(\xi,t) d\xi + 3c^4 t;\\
I_2[u,t] =& c^2;\\
I_3[u,t] =& \int_{-\infty}^{0} \left(-u^2(\xi,t) + c^4\right) d\xi + \int_0^{\infty} \left(-u^2(\xi,t)\right) d\xi - 4c^6 t;\\
I_4[u,t] =& -2c^4;\\
I_5[u,t] =& \int_{-\infty}^{0} \left(2u^3(\xi,t) + u_\xi^2(\xi,t) - 2c^6\right) d\xi\\
&+ \int_0^{\infty} \left(2u^3(\xi,t) + u_\xi^2(\xi,t)\right) d\xi + 9c^8 t;\\
I_6[u,t] =& \frac{16}{3} c^6,
\end{aligned}$$

and the KdV equation can be represented as $\frac{du}{dt} = \frac{d}{dx} \frac{\delta I_5[u,t]}{\delta u}$.

The integrals of motion that do not depend explicitly on time

Note that the integrals of motion with even numbers are constants, and the odd ones depend linearly on time. Therefore it is easy to construct a sequence which does not depend explicitly on time

$$J_1[u] = I_3[u,t] + \frac{4}{3}c^2 \cdot I_1[u,t] = \int_{-\infty}^{0} \left(-u^2(\xi,t) + \frac{4}{3}c^2 u(\xi,t) - \frac{1}{3}c^4 \right) d\xi$$

$$+ \int_{0}^{\infty} \left(-u^2(\xi,t) + \frac{4}{3}c^2 u(\xi,t) \right) d\xi;$$

$$J_2[u] = I_5[u,t] - 3c^4 \cdot I_1[u,t] = \int_{-\infty}^{0} \left(2u^3(\xi,t) - 3c^4 u(\xi,t) + u_\xi^2(\xi,t) + c^6 \right) d\xi$$

$$+ \int_{0}^{\infty} \left(2u^3(\xi,t) - 3c^4 u(\xi,t) + u_\xi^2(\xi,t) \right) d\xi;$$

$$J_3[u] = \int_{-\infty}^{0} \left(-5u^4(\xi,t) - 10u(\xi,t)u_\xi^2(\xi,t) + 8c^6 u(\xi,t) - u_{\xi\xi}^2(\xi,t) - 3c^8 \right) d\xi$$

$$+ \int_{0}^{\infty} \left(-5u^4(\xi,t) - 10u(\xi,t)u_\xi^2(\xi,t) + 8c^6 u(\xi,t) - u_{\xi\xi}^2(\xi,t) \right) d\xi.$$

The integrals of motion in terms of the scattering data

Consider the Cauchy problem for the KdV equation

$$u_t(x,t) = 6u(x,t)u_x(x,t) - u_{xxx}(x,t), \quad u|_{t=0} = u_0(x),$$

and the corresponding spectral Schödinger equation

$$-\frac{d^2}{dx^2}y(x) + u_0(x)y(x) = k^2 y(x), \quad k \in \overline{\mathbb{C}^+}, \quad x \in \mathbb{R},$$

where $u(\cdot,t) \in L^1_{loc}(\mathbb{R})$ and

$$\lim_{x\to-\infty} u(x,t) = c^2, \quad \lim_{x\to+\infty} u(x,t) = 0,$$

also we assume that

$$\int_0^\infty (1+|x|)|u(x,t)| + |u(-x,t) - c^2|)dx < \infty, \quad \forall t \in \mathbb{R}.$$

The integrals of motion in terms of the scattering data

The following facts are valid:

- The equation $-y'' + u_0 y = k^2 y$ has two Jost solutions $\phi(k,x,t)$ and $\phi_1(k,x,t)$ with the asymptotic behavior

$$\lim_{x\to+\infty} \phi(k,x,t)e^{-\mathrm{i}kx} = 1, \quad \mathrm{Im}\ k \geq 0,$$
$$\lim_{x\to-\infty} \phi_1(k,x,t)e^{\mathrm{i}k_1 x} = 1, \quad \mathrm{Im}\ k_1 \geq 0,$$

  where $k_1 = \sqrt{k^2 - c^2}$. These solutions satisfy the scattering relation

$$T(k,t)\phi_1(k,x,t) = \overline{\phi(k,x,t)} + R(k,t)\phi(k,x,t), \quad k \in \mathbb{R},$$

  where $T(k,t), R(k,t)$ are the right transmission and reflection coefficients.

- The spectrum of this operator consists of absolutely continuous part $\mathbb{R}_+$ and a finite number of negative eigenvalues $-\varkappa_1^2 < \ldots < -\varkappa_N^2 < 0$. The continuous spectrum consists of the spectrum of multiplicity one on $[0, c^2]$ and of the spectrum of multiplicity two on $[c^2; \infty)$.

The integrals of motion in terms of the scattering data

- The solutions $\phi(\mathrm{i}\varkappa_l, x, t)$ and $\phi_1(\mathrm{i}\varkappa_l, x, t)$ are linearly dependent eigenfunctions of the Schödinger operator. The corresponding normalizing constants

$$\gamma_l = \Big(\int_{\mathbb{R}} \phi^2(\varkappa_l, x, t)dx\Big)^{-1}, \quad \gamma_{l,1} = \Big(\int_{\mathbb{R}} \phi_1^2(\varkappa_l, x, t)dx\Big)^{-1}.$$

- The following identity is true

$$1 - |R(k,t)|^2 = \frac{k_1}{k}|T(k,t)|^2, \quad k_1 \in \mathbb{R}.$$

- The value $|T(k,t)|$ doesn't depend on t when $k_1 \in \mathbb{R}$, and

$$\arg\ T(k,t) = \arg\ T(k,0) - 4k^3 t, \quad k \in [-c; c].$$

  Moreover, $\gamma_l(t) = \gamma_l(0)e^{8\varkappa_l^3 t}$.

- The solution $u(x,t)$ the initial value problem for the KdV equation with steplike initial data can be uniquely restored by the right scattering data

$$\mathcal{S}(t) = \{R(k,t),\ k \in \mathbb{R}; \quad -\varkappa_l^2; \quad \gamma_l(t), \quad l = \overline{1, N}\}.$$

- If $u(\cdot, t) \in \Sigma$, then $R(k,t) = O(\frac{1}{k^{n+1}}),\ k \to \infty, \forall n$.

The integrals of motion in terms of the scattering data

Problem 2

To express the regularized integrals of motion in terms of the scattering data.

Introduce the function

$$B(k,t) := \frac{\mathrm{i}}{\sqrt{k^2-c^2}} \log\Big[\prod_{l=1}^{N}\frac{k-\mathrm{i}\varkappa_l}{k+\mathrm{i}\varkappa_l}\,T(k,t)\sqrt{\frac{k_1}{k}}\Big].$$

where $T(k,t)$ is the transmission coefficient. The function $T(k,t)$ has poles $\mathrm{i}\varkappa_l$, $l=\overline{1,N}$ in the upper half plane, so the function $B(k,t)$ is holomorphic for $\mathrm{Im}\, k>0$ and does not have zeros there. At $\mathrm{Im}\, k>0$ by Cauchy's theorem we have the integral representation

$$B(k,t) = \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{Im}\, B(s,t)}{s-k}\,ds.$$

where

$$\mathrm{Im}\, B(k,t) = \begin{cases} \frac{1}{2\sqrt[+]{k^2-c^2}}\log(1-|R(k,t)|^2), & k\in\mathbb{R}\backslash[-c;c], \\ \frac{1}{\sqrt[+]{c^2-k^2}}\Big[\frac{1}{2}\arg R(k,t)+\frac{1}{2}\arg\frac{\sqrt{k^2-c^2}}{k}+\sum_{l=1}^{N}\arg\frac{k-\mathrm{i}\varkappa_l}{k+\mathrm{i}\varkappa_l}\Big], & k\in[-c;c]. \end{cases}$$

The integrals of motion in terms of the scattering data

Problem 2

To express the regularized integrals of motion in terms of the scattering data.

Expand the function $\log\ T^{-1}(k,t)$ with respect to $(-2\mathrm{i}k)^n$:

$$\log\ T^{-1}(k,t) = \sum_{n=1}^{\infty} \frac{\mathcal{C}_n(t)}{(-2\mathrm{i})^n} \cdot \frac{1}{k^n}.$$

We get the following coefficients $\mathcal{C}_n(t)$:

$$\mathcal{C}_{2j-1}(t) = \frac{1}{\pi} \int_{\mathbb{R}\setminus[-c;c]} \log\left(1 - |R(s,t)|^2\right) \frac{(-1)^{j-1} \cdot 2^{2j-2} \cdot d_{2j-1}(s)}{\sqrt{s^2 - c^2}} ds$$

$$+ \frac{1}{\pi} \int_{-c}^{c} \left(\frac{1}{2} \arg R(s,t) + \frac{1}{2} \arg \frac{\sqrt{s^2 - c^2}}{s} + \sum_{l=1}^{N} \arg \frac{s - \mathrm{i}\varkappa_l}{s + \mathrm{i}\varkappa_l}\right) \frac{(-1)^{j-1} \cdot 2^{2j-1} \cdot d_{2j-1}(s)}{\sqrt{c^2 - s^2}} ds$$

$$- \frac{2^{2j}}{2j-1} \sum_{l=1}^{N} \varkappa_l^{2j-1}, \qquad \mathcal{C}_{2j} = \frac{(-1)^{j-1} \cdot (2c)^{2j}}{4j}, \qquad j = 1, 2, \ldots$$

here the functions $d_{2j-1}(s), j = 1, 2, \ldots$ can be found from the following relations

$$d_{2j-1}(s) = a_1(s) \cdot b_j + \ldots + a_j(s) \cdot b_1,$$

where $a_p(s),\ b_p,\ p = \overline{1, j}$:

$$a_p(s) = s^{2p-1}, \quad b_p = \frac{(2p-2)!}{(3-2p)(p-1)!^2 4^{p-1}} c^{2(p-1)}.$$

The integrals of motion in terms of the scattering data

Result 2 (K.Andreiev'16 to appear)

$$J_1[u] = \mathcal{C}_3(t) + \frac{4}{3}c^2 \cdot \mathcal{C}_1(t) = \frac{1}{\pi}\int_{\mathbb{R}\setminus[-c;c]} \log\left(1 - |R(s,t)|^2\right) \frac{-4s^3 + \frac{10}{3}c^2 s}{\sqrt{s^2 - c^2}} ds$$

$$+ \frac{1}{\pi}\int_{-c}^{c} \left(\frac{1}{2}\arg R(s,t) + \frac{1}{2}\arg\frac{\sqrt{s^2 - c^2}}{s} + \sum_{l=1}^{N}\arg\frac{s - \mathrm{i}\varkappa_l}{s + \mathrm{i}\varkappa_l}\right) \frac{-8s^3 + \frac{20}{3}c^2 s}{\sqrt{c^2 - s^2}} ds$$

$$+ \sum_{l=1}^{N}\left(-\frac{16}{3}\varkappa_l^3 + \frac{16}{3}c^2\varkappa_l\right),$$

$$J_2[u] = \mathcal{C}_5(t) - 3c^4 \cdot \mathcal{C}_1(t) = \frac{1}{\pi}\int_{\mathbb{R}\setminus[-c;c]} \log\left(1 - |R(s,t)|^2\right) \frac{16s^5 - 8c^2s^3 - 5c^4 s}{\sqrt{s^2 - c^2}} ds$$

$$+ \frac{1}{\pi}\int_{-c}^{c} \left(\frac{1}{2}\arg R(s,t) + \frac{1}{2}\arg\frac{\sqrt{s^2 - c^2}}{s} + \sum_{l=1}^{N}\arg\frac{s - \mathrm{i}\varkappa_l}{s + \mathrm{i}\varkappa_l}\right) \frac{32s^5 - 16c^2s^3 - 10c^4 s}{\sqrt{c^2 - s^2}} ds$$

$$+ \sum_{l=1}^{N}\left(-\frac{64}{5}\varkappa_l^5 + 9c^4\varkappa_l\right).$$

The right hand side of these formulas do not depend on t, since:

$$|R(s,t)| = |R(s,0)|, \quad \frac{1}{2}\arg R(s,t) = \frac{1}{2}R(s,0) + 4\mathrm{i}s^3 t.$$

The symplectic form in the steplike case

Let

$$\mathcal{S}_i = \{R_i(k),\ k \in \mathbb{R};\ \varkappa_l^{(i)};\ \gamma_l^{(i)},\ l = \overline{1, N_i}\}, \quad i = 1, 2$$

be the scattering data for the two potentials $u_1(x)$ and $u_2(x)$. Then the following equation for $K(x, y)$ is valid

$$K(x, y) + F(x, y) + \int_x^\infty K(x, \xi) F(\xi, y) d\xi = 0, \quad y > x,$$

where the kernel $F(x, y)$ has the form

$$F(x, y) = \frac{1}{2\pi} \int_{-\infty}^\infty (R_2(k) - R_1(k))\phi(x, k)\phi(y, k) dk +$$

$$+ \sum_{l=1}^{N_2} \gamma_l^{(2)} \phi(x, \mathrm{i}\varkappa_l^{(2)})\phi(y, \mathrm{i}\varkappa_l^{(2)}) - \sum_{l=1}^{N_1} \gamma_l^{(1)} \phi(x, \mathrm{i}\varkappa_l^{(1)})\phi(y, \mathrm{i}\varkappa_l^{(1)}).$$

here and next $\phi(x, k)$ is the Jost solution that corresponds to the potential u_1. This equation has the unique solution, and

$$u_2(x) - u_1(x) = 2\frac{d}{dx} K(x, x).$$

[6] *Kay J., Moses H.E.* The determination of the scaterring potential from the spectral measure function, III, Nuovo Cimento 3, No.2 (1956), 277-304.

The symplectic form in the steplike case

Consider the potential u on the manifold Σ, then $\delta u \in \Sigma_0$. Introduce the corresponding symplectic form

$$\Omega(\delta_1 u, \delta_2 u) = \int_{-\infty}^{\infty} dx \int_{x}^{\infty} dy(\delta_1 u(x) \cdot \delta_2 u(y) - \delta_1 u(y) \cdot \delta_2 u(x)).$$

Problem 3

To express the symplectic form in terms of the scattering data.

Let $\delta_1 u(x), \delta_2 u(x)$ be two variations of the potential $u(x)$, and $\mathcal{S}, \delta_1 \mathcal{S}, \delta_2 \mathcal{S}$ are the corresponding scattering data and their variations. Then we get the variation $\delta u(x)$ in terms of scattering data

$$\delta u(x) = -\frac{d}{dx}\left[\frac{1}{\pi}\int_{-\infty}^{\infty} \delta R(k)\phi^2(x,k)dk + 2\sum_{k=1}^{N}\left(\phi_l^2(x)\delta\gamma_l + 2\mathrm{i}\gamma_l\phi_l(x)\dot{\phi}_l(x)\delta\varkappa_l\right)\right]$$

where

$$\phi_l(x) = \phi(x, \mathrm{i}\varkappa_l), \quad \dot{\phi}_l(x) = \frac{d}{dk}\phi(x,k)|_{k=\mathrm{i}\varkappa_l}.$$

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After substituting this into the Marchenko equation and grouping summands we get

$$\Omega(\delta_1 u, \delta_2 u) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(k,q)\left[\delta_1 R(k)\delta_2 R(q) - \delta_1 R(q)\delta_2 R(k)\right] dkdq$$

$$+ \sum_{l=1}^{N}\int_{-\infty}^{\infty} B_l(k)[\delta_1 R(k)\delta_2 \varkappa_l - \delta_1 \varkappa_l \delta_2 R(k)]dk$$

$$+ \sum_{i,j=1}^{N}[C_{ij}(\delta_1 \varkappa_i \delta_2 \gamma_j - \delta_1 \gamma_j \delta_2 \varkappa_i) + D_{ij}(\delta_1 \varkappa_i \delta_2 \varkappa_j - \delta_1 \varkappa_j \delta_2 \varkappa_i)]$$

$$+ \sum_{l=1}^{N}\int_{-\infty}^{\infty} E_l(k)(\delta_1 R(k)\delta_2 \gamma_l - \delta_1 \gamma_l \delta_2 R(k))dk + \sum_{l,j=1}^{N} F_{i,j}(\delta_1 \gamma_i \delta_2 \gamma_j - \delta_1 \gamma_j \delta_2 \gamma_i)$$

where

$$A(k,q) = \frac{1}{8\pi^2}\int_{-\infty}^{\infty} < \phi^2(x,k), \phi^2(x,q) > dx, \quad B_l(k) = \frac{\mathrm{i}\gamma_l}{\pi}\int_{-\infty}^{\infty}\left\langle \phi^2(x,k), \phi_l(x)\dot{\phi}_l(x)\right\rangle dx,$$

$$C_{ij} = 2\mathrm{i}\gamma_i \int_{-\infty}^{+\infty}\left\langle \phi_i(x)\dot{\phi}_i(x), \phi_j^2(x)\right\rangle dx,$$

$$D_{ij} = -4\gamma_i\gamma_j \int_{-\infty}^{+\infty}\left\langle \phi_i(x)\dot{\phi}_i(x), \phi_j(x)\dot{\phi}_j(x)\right\rangle dx,$$

$$E_l(k) = \frac{2}{\pi}\int_{-\infty}^{\infty} < \phi^2(x,k), \phi_l^2(x) > dx, \quad F_{i,j} = 2\int_{-\infty}^{\infty} < \phi_i^2(x), \phi_j^2(x) > dx,$$

here

$$< f(x), g(x) > = \frac{d}{dx}f(x)\cdot g(x) - \frac{d}{dx}g(x)\cdot f(x).$$

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Result 3

We get the symplectic form in terms of the scattering data, but in NON-canonical form

$$\Omega_{\mathcal{S}}(\delta_1\mathcal{S},\delta_2\mathcal{S}) = \int_{\mathbb{R}\setminus[-c;c]} \frac{ik}{\pi}\frac{|R(k)|^2}{1-|R(k)|^2}\Big(\delta_1 R(k)\delta_2 R(q) - \delta_1 R(q)\delta_2 R(k)\Big)dk$$
$$+\frac{2P}{\pi^2}\int_{\mathbb{R}\setminus[-c;c]}\int_{\mathbb{R}\setminus[-c;c]} \frac{k_1^2+q_1^2}{k^2-q^2}\frac{kq}{k_1q_1}\frac{R(-k)}{1-|R(k)|^2}\frac{R(-q)}{1-|R(q)|^2}\Big(\delta_1 R(k)\delta_2 R(q) - \delta_1 R(q)\delta_2 R(k)\Big)dkdq$$
$$+\frac{4}{\pi}\sum_{l=1}^{N}\int_{\mathbb{R}\setminus[-c;c]}\frac{\varkappa_l^2+c^2-k_1^2}{k^2+\varkappa_l^2}\frac{k}{k_1}\frac{R(-k)}{1-|R(k)|^2}\Big(\delta_1 R(k)\delta_2\varkappa_l - \delta_1\varkappa_l\delta_2 R(k)\Big)dk$$
$$+2\sum_{l=1}^{N}\frac{\varkappa_l}{\gamma_l}\Big(\delta_1\varkappa_l\delta_2\gamma_l - \delta_1\gamma_l\delta_2\varkappa_l\Big) + 2\sum_{i,j=1,i\neq j}^{N}\frac{\varkappa_i^2+\varkappa_j^2}{\varkappa_j^2-\varkappa_i^2}\Big(\delta_1\varkappa_l\delta_2\gamma_l - \delta_1\gamma_l\delta_2\varkappa_l\Big).$$

Suggestion

The canonical variables for this form are like

$$P(k,t) = \begin{cases} -\frac{k}{\pi}\log(1-|R(k,t)|^2), & k\in\mathbb{R}\setminus[-c,c],\\ |R(k,t)|, & k\in[-c,c]. \end{cases}$$
$$Q(k,t) = \arg T(k,t) - \arg R(k,t), \quad k\in\mathbb{R}.$$

Thank you for your attention!