Spectral Theory, Differential Equations and Probability
Johannes Gutenberg Universitat Mainz
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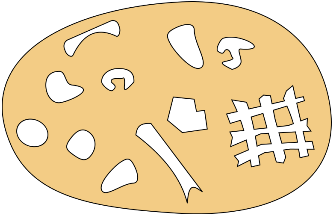
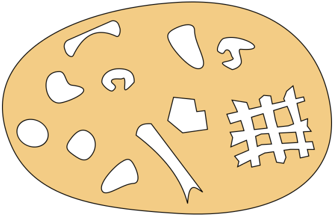
Spectral problems for elliptic operators in strongly perforated domains

Andrii Khrabustovskyi

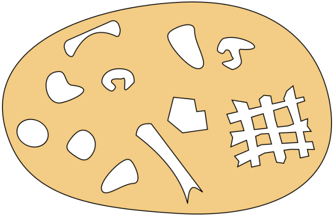
Institute for Analysis
Karlsruhe Institute of Technology





Outline of the talk

1. Homogenization in perforated domains: an overview
    - Dirichlet problem
    - Neumann problem



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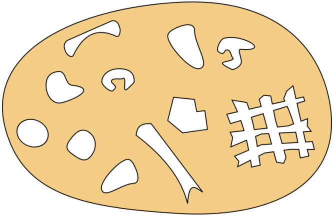

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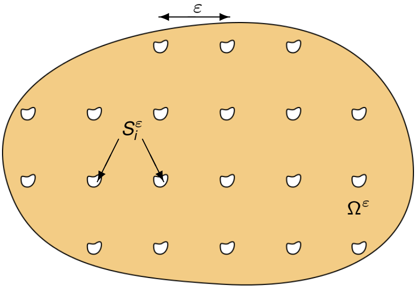
   - Neumann problem
2. Homogenization in domains with traps
3. Applications: periodic operators with predefined spectral gaps



Domain Ω^ε

$$\Omega^\varepsilon = \Omega \setminus S^\varepsilon, \quad S^\varepsilon = \bigcup_i S_i^\varepsilon$$

- $\Omega \subset \mathbb{R}^n$ is a bounded domain
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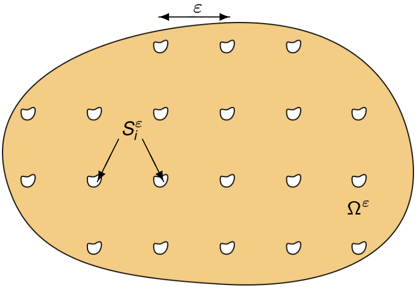

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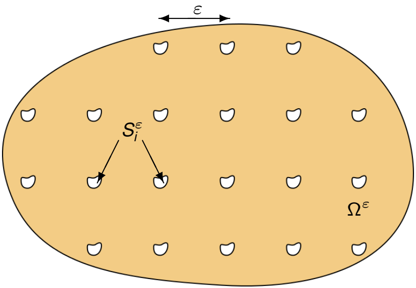
Operator $\mathcal{A}_D^\varepsilon$

We denote by $\mathcal{A}_D^\varepsilon$ the **Dirichlet Laplacian** in Ω^ε.

Its spectrum is purely discrete: $\sigma(\mathcal{A}_D^\varepsilon) = \left\{\lambda_D^{\varepsilon,k} : \ k \in \mathbb{N}\right\}, \ \lambda_D^{\varepsilon,k} \underset{k\to\infty}{\nearrow} \infty$



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Our goal: to describe the behaviour of $\sigma(\mathcal{A}_D^\varepsilon)$ as $\varepsilon \to 0$.

Critical regime is determined by $$\alpha := \lim_{\varepsilon\to 0} \varepsilon^{-n}\begin{cases}(\eta^\varepsilon)^{n-2}, & n>2\\ |\ln \eta^\varepsilon|^{-1}, & n=2\end{cases}$$

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Theorem: $\alpha < \infty$

One has for each $f \in L_2(\Omega)$:

$$(\mathcal{A}_D^\varepsilon + I)^{-1} f \overset{L_2}{\to} (\mathcal{A}_D + I)^{-1} f \quad \text{as } \varepsilon \to 0,$$

where the operator $\mathcal{A}_D$ is as follows: $\mathrm{dom}(\mathcal{A}_D) = H^2(\Omega) \cap H_0^1(\Omega)$,

$$\mathcal{A}_D u = -\Delta u + Vu, \text{ where } V = \alpha C_S.$$

For $S = \{|x| < 1\}$ one has $C_S = \begin{cases} 2\pi, & n=2, \\ (n-2)|\partial S|, & n \geq 3. \end{cases}$

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Remark: non-periodic perforations

[KHRUSLOV, 1971], [BUTTAZZO–DAL MASO–MOSCO, 1987]

In the “Dirichlet” case the form of the limiting operator is independent of the removed domain S^ε: it is always of the form

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where V is a certain distribution.

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Theorem: $\alpha < \infty$

$\sigma(\mathcal{A}_D^\varepsilon)$ converges to $\sigma(\mathcal{A}_D)$ in the Hausdorff sense as $\varepsilon \to 0$, i.e.

(i) If a family $\left\{\lambda^\varepsilon \in \sigma(\mathcal{A}_D^\varepsilon)\right\}_\varepsilon$ has a convergent subsequence, namely, $\lambda^\varepsilon \to \lambda$ as $\varepsilon = \varepsilon_k \to 0$, then $\lambda \in \sigma(\mathcal{A}_D)$.

(ii) If $\lambda \in \sigma(\mathcal{A}_D)$ then there exists a family $\left\{\lambda^\varepsilon \in \sigma(\mathcal{A}_D^\varepsilon)\right\}_\varepsilon$ such that $\lim\limits_{\varepsilon\to 0} \lambda^\varepsilon = \lambda$.

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Moreover, for each $k \in \mathbb{N}$

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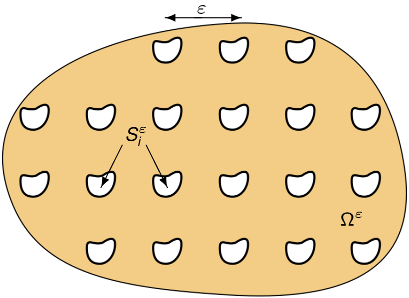
Theorem: $\alpha = \infty$

One has: $\lambda_D^{\varepsilon,1} \to \infty$ as $\varepsilon \to 0$.



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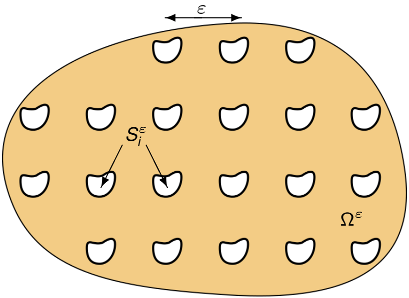

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Operator $\mathcal{A}_N^\varepsilon$

We denote by $\mathcal{A}_N^\varepsilon$ the **Neumann Laplacian** in Ω^ε.

Its spectrum is purely discrete: $\sigma(\mathcal{A}_N^\varepsilon) = \left\{\lambda_N^{\varepsilon,k} : k \in \mathbb{N}\right\}$, $\lambda_N^{\varepsilon,k} \underset{k\to\infty}{\nearrow} \infty$.

The case $\eta^\varepsilon = \varepsilon$

Recall that $S_i^\varepsilon = \eta^\varepsilon S + \varepsilon i$. We assume that $S \subset Y := (0,1)^n$.

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Limiting operator $\mathcal{A}_N$ has the form

$$\mathcal{A}_N = -\frac{1}{\widehat{b}}\mathrm{div}(\widehat{A}\nabla).$$

Here $\widehat{A}$ is a positive-definite matrix with entries defined by

$$\widehat{A}_{ij} = \int\limits_{Y\setminus S} (\nabla w_i + e_i)\cdot(\nabla w_j + e_j)\mathrm{d}x, \quad \widehat{b} = |Y \setminus S|,$$

where w_i is a solution to the problem

$$\begin{aligned} -\Delta w_i = 0 &\quad \text{in } Y \setminus S \\ \frac{\partial w_i}{\partial n} + e_i \cdot n = 0 &\quad \text{on } \partial S \\ w_i &\quad \text{is } Y\text{-periodic} \end{aligned}$$

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The case $\eta^\varepsilon \ll \varepsilon$

Limiting operator $\mathcal{A}_N$ is the Neumann Laplacian in Ω

- Periodic perforations
  - D. Cioranescu, J. Saint Jean Paulin, J. Math. Anal. Appl. 71 (1979)
  - E. Acerbi, V. Chiadó Piat, G. Dal Maso, D. Percivale, Nonlinear Anal. 18 (1992)
  - G. Allaire, F. Murat, Asymptotic Anal. 7 (1993)
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- Non-periodic perforations
  - E.Ya. Khruslov, Math. Sbornik 35 (1978)
  - G. Nguetseng, J. Math. Anal. Appl. 289 (2004)

Key ingredient of the proof

One can construct an extension operator $\Pi^\varepsilon : H^1(\Omega^\varepsilon) \to H^1(\Omega)$ satisfying

(a) $\Pi^\varepsilon u = u$ in Ω^ε

(b) $\|\Pi^\varepsilon u\|_{H^1(\Omega)} \leq C\|u\|_{H^1(\Omega^\varepsilon)}$, C is independent of u and ε

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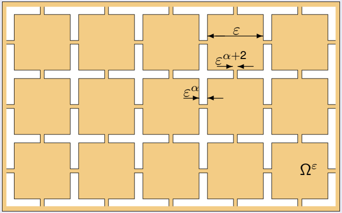
Theorem (Acerbi–Chiadó Piat–Dal Maso–Percivale, 1992)

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let D be a **connected** $\mathbb{Z}^n$-periodic domain with Lipschitz boundary, we set $D^\varepsilon = \varepsilon D$.

Then the family $\{\Omega^\varepsilon := \Omega \cap D^\varepsilon\}_\varepsilon$ is strongly connected.

Example: the family $\{\Omega^\varepsilon\}_\varepsilon$ which is **not** strongly connected



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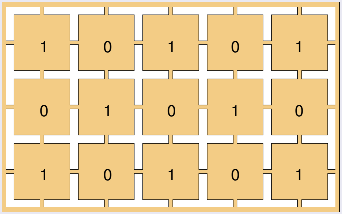

Proof. Suppose that Ω^ε is strongly connected. Then there exists operator Π^ε satisfying properties (a)-(b). We take the following function u^ε:

$$u^\varepsilon = \begin{cases} 1 \vee 0, & \text{in squares} \\ \text{linear}, & \text{otherwise} \end{cases}$$

One has $\|u^\varepsilon\|_{H^1(\Omega^\varepsilon)} \leq C$, whence $\|\Pi^\varepsilon u^\varepsilon\|_{H^1(\Omega)} \leq C_1$. Then $\exists u \in H^1(\Omega)$:

$$\|u^\varepsilon - u\|_{L_2(\Omega^\varepsilon)} \leq \|\Pi^\varepsilon u^\varepsilon - u\|_{L_2(\Omega)} \to 0 \text{ as } \varepsilon = \varepsilon_k \to 0.$$

This contradicts to the structure of the function u^ε.

If $\{\Omega^\varepsilon\}_\varepsilon$ is **strongly connected** then the limiting operator has the form

$$\mathcal{A}_N = -\frac{1}{\widehat{b}(x)}\mathrm{div}(\widehat{A}(x)\nabla),$$

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If $\{\Omega^\varepsilon\}_\varepsilon$ is **weakly connected**, namely

$$\Omega^\varepsilon = \Omega_1^\varepsilon \cup \Omega_2^\varepsilon \cup \dots \cup \Omega_m^\varepsilon \cup \omega^\varepsilon,$$

where Ω_k^ε, $k = 1, \dots, m$ are strongly connected, $|\omega^\varepsilon| \to 0$, then the limiting operator is **multi-component**.

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$$\mathcal{A}_N = \begin{bmatrix} -\frac{1}{\widehat{b}_1(x)}\mathrm{div}(\widehat{A}_1(x)\nabla) + C_{11}(x) & C_{12}(x) \\ C_{21}(x) & -\frac{1}{\widehat{b}_2(x)}\mathrm{div}(\widehat{A}_2(x)\nabla) + C_{22}(x) \end{bmatrix}$$

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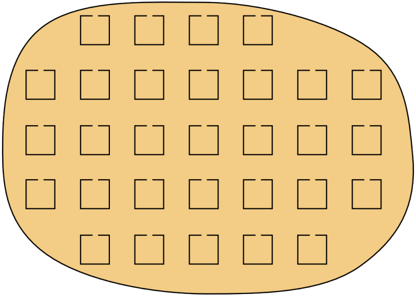
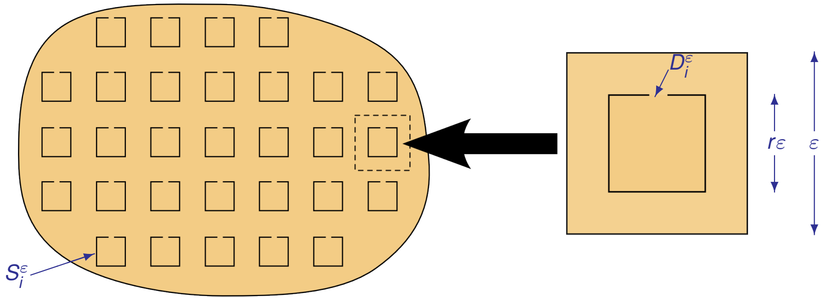
- M. Briane, Ric. Mat. 47(1998); Ann. Mat. Pur. Appl. 177(1999)





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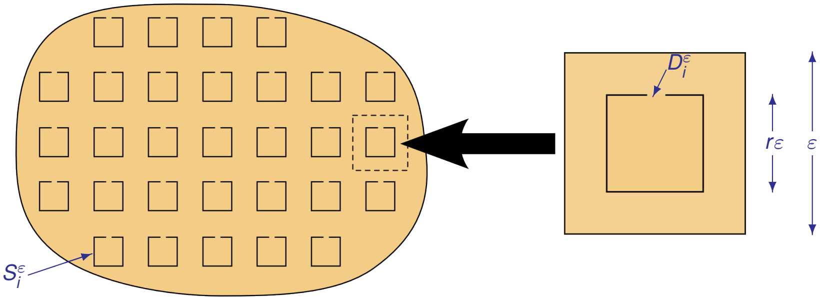

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The holes are very small, namely $D_i^\varepsilon \simeq \eta^\varepsilon D$, where $D \subset \mathbb{R}^{n-1}$, and the following limit is **positive**:

$$\alpha := \lim_{\varepsilon \to 0} \varepsilon^{-n} \begin{cases} (\eta^\varepsilon)^{n-2}, & n > 2 \\ |\ln \eta^\varepsilon|^{-1}, & n = 2 \end{cases}$$

Neumann Laplacian in Ω^ε

We denote by $\mathcal{A}$ the Neumann Laplacian in Ω^ε, i.e. the operator acting in $L_2(\Omega^\varepsilon)$ and associated with the sesquilinear form $\mathfrak{a}$,

$$\mathrm{dom}(\mathfrak{a}) = H^1(\Omega^\varepsilon), \qquad \mathfrak{a}[u, v] = \int\limits_{\Omega^\varepsilon} \nabla u \cdot \overline{\nabla v} \mathrm{d}x.$$

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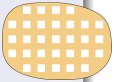
The limiting operator $\mathcal{A}$

$$\mathcal{A} = \begin{pmatrix} -\frac{1}{\widehat{b}}\mathrm{div}\left(\widehat{A}\nabla\right) + p & -p \\ -q & q \end{pmatrix},$$

$$\mathrm{dom}(\mathcal{A}) = H^2(\Omega) \times L_2(\Omega)$$

- $\widehat{b} = 1 - r^n$, $\widehat{A}$ is the effective conductivity matrix for 
- $q = \dfrac{\alpha C_s}{r^n}$, $\qquad p = \dfrac{q r^n}{1 - r^n}$

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Simple observation: the problem $\mathcal{A}U=\lambda U$ is equivalent to

$$-\mathrm{div}\left(\widehat{A}\nabla u\right)=\lambda\left(1+\frac{p}{q-\lambda}\right)u$$

The limiting operator $\mathcal{A}$

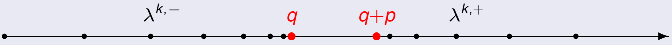
$$\mathcal{A} = \begin{pmatrix} -\mathrm{div}\left(\widehat{A}\nabla\right) + p & -p \\ -q & q \end{pmatrix}, \qquad \mathrm{dom}(\mathcal{A}) = H^2(\mathbb{R}^n) \times L_2(\mathbb{R}^n)$$

Simple observation: the problem $\mathcal{A}U = \lambda U$ is equivalent to

$$-\mathrm{div}\left(\widehat{A}\nabla u\right) = \lambda\left(1 + \frac{p}{q-\lambda}\right)u$$

Lemma: the spectrum of $\mathcal{A}$

$$\sigma_{disc}(\mathcal{A}) = \left\{\lambda^{k,-} :\ k \in \mathbb{N}\right\} \cup \left\{\lambda^{k,+} :\ k \in \mathbb{N}\right\}, \quad \sigma_{ess}(\mathcal{A}) = \{q\}.$$



$$\lambda^{1,-} = 0, \quad \lambda^{k,-} \nearrow q, \quad \lambda^{1,+} = q+p, \quad \lambda^{k,+} \nearrow \infty.$$

Theorem

$\sigma(\mathcal{A}^\varepsilon)$ converges to $\sigma(\mathcal{A})$ in the Hausdorff sense as $\varepsilon \to 0$.

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Remark 1

Theorem below remains valid if $\Omega = \mathbb{R}^n$. In this case

$$\sigma(\mathcal{A}) = [0,\infty) \setminus (q, q+p).$$

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$$\sigma(\mathcal{A}) = [0,\infty) \setminus (q, q+p).$$

Remark 2

We assume that η^ε depends on a period cell, namely $\eta^\varepsilon = \eta_i^\varepsilon$ and

$$n > 2: \qquad \lim_{\varepsilon\to 0} \sum_i (\eta_i^\varepsilon)^{n-2} \delta(\cdot - x_i^\varepsilon) = \alpha \text{ in } D'(\Omega),$$
$$n = 2: \qquad \lim_{\varepsilon\to 0} \sum_i |\ln \eta_i^\varepsilon|^{-1} \delta(\cdot - x_i^\varepsilon) = \alpha \text{ in } D'(\Omega),$$

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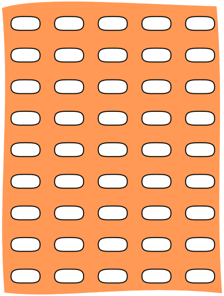
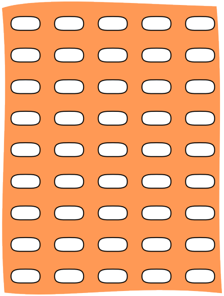
where $\alpha \in C(\overline{\Omega})$. Then Theorem below remains, but now q and p are continuous functions. In particular,

$$\sigma_{ess}(\mathcal{A}) = [\min_{x\in\overline{\Omega}} q(x), \max_{x\in\overline{\Omega}} q(x)].$$

- Linear parabolic equations
  - E.YA. KHRUSLOV, Dokl. Akad. Nauk SSSR 309 (1989)
  - E.YA. KHRUSLOV, Progr. Nonlinear Differ. Equat. Appl. 5 (1991)
- Variational problems
  - L.S. PANKRATOV, J. Soviet Math. 58 (1992)
- Semi-linear parabolic equations
  - A. BOURGEAT, L. PANKRATOV, Appl. Anal. 64 (1997)
- Non-linear elliptic equations
  - A. KOVALEVSKII, Ukr. Mat. Zh. 47 (1995)
- Spectral properties
  - V.A.MARCHENKO, E.YA.KHRUSLOV, Homogenization of PDEs, 2006
  - A.K., E. KHRUSLOV, Math. Methods Appl. Sci. 38 (2014)
  - A.K., J. Math. Phys. 55 (2014)



The problem we are going to study relates to the spectral theory of elliptic self-adjoint differential operators with **periodic** coefficients in unbounded domains with some **periodic** structure.



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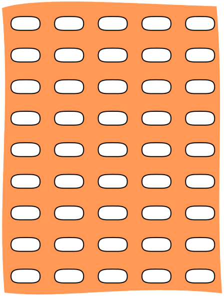
It is known that the spectra of such operators have a band structure, i.e. the spectrum is a locally finite union of compact intervals called **bands**.



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The non-empty interval (α, β) is called a **gap** in the spectrum of the operator $\mathcal{A}$ if

$$(\alpha, \beta) \cap \sigma(\mathcal{A}) = \varnothing \text{ and } \alpha, \beta \in \sigma(\mathcal{A}).$$

In general the presence of gaps in the spectrum is not guaranteed!

Example: $\sigma(-\Delta_{\mathbb{R}^n}) = [0, \infty)$.

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Problem 1

For a given class $\mathcal{P}$ of periodic differential operators, construct an operator $\mathcal{A} \in \mathcal{P}$ with at least one gap in the spectrum

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In what follows, we are interested in the class $\mathcal{P}$ consisting of Neumann Laplacians in periodic domains.

We denote by $\mathcal{A}_\Omega$ the Laplacian in the domain $\Omega \subset \mathbb{R}^n$ subject to Neumann or Dirichlet boundary conditions.

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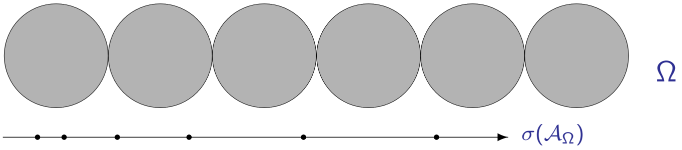
- Large gaps versus small bands

Post (2003); Pankrashkin (2010); Nazarov (2009, 2010); Borisov (2015)



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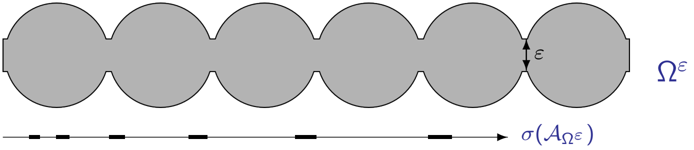

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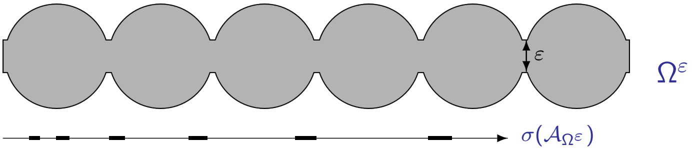

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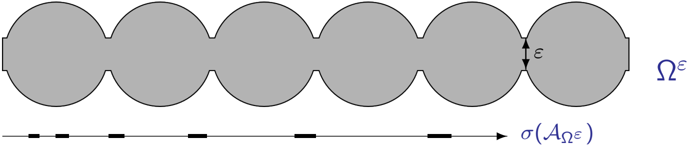

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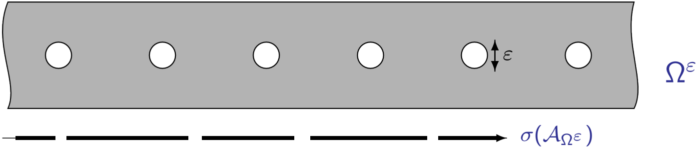
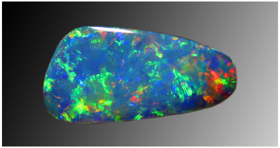


Photonic crystals are periodic nanostructures that have been attracting much attention in recent years. Their main feature is that the electromagnetic waves of certain frequencies fail to propagate in them, which is caused by gaps in the spectrum of underlying Maxwell operators (or related scalar operators).



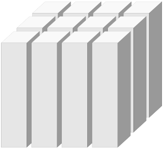


Figure 1: Opal – an example of natural photonic crystal (photo: https://en.wikipedia.org/wiki/Opal).

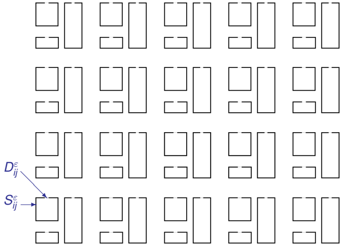
Figure 2: The dielectric constant equals 1 on the vertical columns, and it equals $\gg 1$ in the rest of the media.

- A. Figotin, P. Kuchment, SIAM J. Appl. Math. 56 (1996).
- P. Kuchment, The mathematics of photonic crystals. In: Mathematical Modeling in Optical Science, Frontiers in Applied Mathematics 22, SIAM, 2001.
- W. Dörfler, A. Lechleiter, M. Plum, G. Schneider, C. Wieners, Photonic Crystals. Mathematical Analysis and Numerical Approximation, Springer, Berlin, 2011.

Problem 2

Construct an operator $\mathcal{A} \in \mathcal{P}$ having gaps which are close (in some natural sense) to predefined intervals

[A.K., Journal of Mathematical Physics 55 (2014)]



$$\Omega^{\varepsilon} = \mathbb{R}^n \setminus \left(\bigcup_{i\in\mathbb{Z}^n} \bigcup_{j=1}^{m} S_{ij}^{\varepsilon} \right)$$

The holes D_{ij}^{ε} are of the diameter

$$d_j^{\varepsilon} = \begin{cases} d_j \varepsilon^{\frac{n}{n-2}}, \\ \exp\left(-\frac{1}{d_j \varepsilon^2}\right). \end{cases}$$

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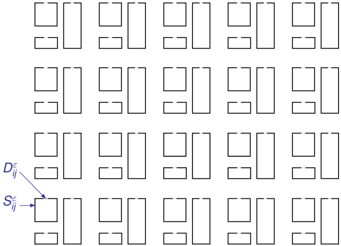

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Results

- the Neumann Laplacian in Ω^ε has at least m gaps as $\varepsilon \ll 0$

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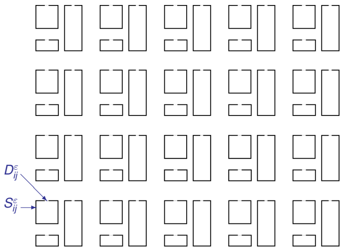

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- the Neumann Laplacian in Ω^ε has at least m gaps as $\varepsilon \ll 0$
- the first m gaps converge as $\varepsilon \to 0$ to some intervals, whose closures are pairwise disjoint; the next gaps (if any) go to ∞,

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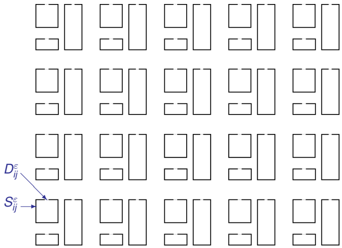

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- one can control the location of these intervals via a suitable choice of d_j and S_{ij}^ε (in fact, only the volume of the domain enclosed by S_{ij}^ε is important, while its shape plays no role)

Papers concerning media with traps

- Elliptic operators in divergence form
  - E.YA. KHRUSLOV, Russian Math. Surveys 45 (1990)
  - L.S. PANKRATOV, I.D. CHUESHOV, Sb. Math. 190 (1999)
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  - A.K., Appl. Analysis 87 (2008)
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  - A.K., Appl. Analysis 87 (2008)
  - A.K., J. Differ. Equations 252 (2012)
- Schrödinger operators with point interactions; quantum graphs
  - D. Barseghyan, A.K., J. Phys. A: Math. Theor. 48 (2015)
  - P. Exner, A.K., J. Phys. A: Math. Theor. 48 (2015)

Thank you for your attention!