Symmetric α-stable distributions for noninteger $\alpha > 2$ and associated stochastic processes.

M.V.Platonova

St. Petersburg State University, Chebyshev laboratory.

Mainz, 04.09.2016-15.09.2016

Introduction

We consider evolution equations

$$\frac{\partial u}{\partial t} = c_\alpha \mathcal{D}_+^\alpha u, \tag{1}$$

$$\frac{\partial u}{\partial t} = c_\alpha \mathcal{D}_-^\alpha u, \tag{2}$$

where $c_\alpha = (-1)^{[\frac{\alpha}{2}]}\Gamma(-\alpha)$ and $\mathcal{D}_\pm^\alpha$ are fractional derivative operators of the order $\alpha > 0$, defined by

$$(\mathcal{D}_\pm^\alpha f)(x) = \frac{1}{\Gamma(-\alpha)} \int_0^\infty \frac{f(x \mp t) - \sum_{k=0}^{[\alpha]} \frac{f^{(k)}(x)}{k!}(\mp t)^k}{t^{1+\alpha}} dt.$$

We also consider an evolution equation

$$\frac{\partial u}{\partial t} = c_\alpha \mathcal{D}^\alpha u, \ \alpha \notin \mathbb{N}, \tag{3}$$

where $\mathcal{D}^\alpha$ is a symmetric fractional derivative operator of the order $\alpha > 0$, defined by

$$\mathcal{D}^\alpha = \mathcal{D}^\alpha_+ + \mathcal{D}^\alpha_-,$$

and therefore

$$(\mathcal{D}^\alpha f)(x) = \frac{1}{\Gamma(-\alpha)} \int\limits_{-\infty}^{+\infty} \frac{f(x-t) - \sum\limits_{k=0}^{[\frac{\alpha}{2}]} \frac{f^{(2k)}(x)}{2k!} t^{2k}}{|t|^{1+\alpha}} dt.$$

For (1), (2) and (3) we consider the Cauchy problem

$$u(0,x) = \varphi(x), \tag{4}$$

where $\varphi \in L_2(\mathbb{R})$.

The case $\alpha \in (0,1) \cup (1,2)$

If $\alpha \in (0,1) \cup (1,2)$, then the solutions (1), (4) and (2), (4) can be represented in the forms

$$u\,(t,x) = \mathbf{E}\varphi(x - \xi_\alpha^+(t)),\; u(t,x) = \mathbf{E}\varphi(x - \xi_\alpha^-(t)), \qquad (5)$$

where $\xi_\alpha^\pm(t)$ are the Lévy processes with the Lévy measure $\Lambda^\pm(dx) = \frac{C_\alpha dx}{|x|^{1+\alpha}} \mathbf{1}_{\mathbb{R}_\pm}(x)$.

The solution (3), (4) can be represented in the form

$$u(t,x) = \mathbf{E}\varphi(x - \xi_\alpha(t)), \qquad (6)$$

where $\xi_\alpha(t)$ is the symmetric stable Lévy process with the Lévy measure $\Lambda(dx) = \frac{C_\alpha dx}{|x|^{1+\alpha}}$.

For $\alpha > 2$ the solutions can not be represented in this form because the fundamental solutions of (1), (2) and (3) are not probability densities.

Previous results.

E.Orsingher, B. Toaldo, 2014 - theory of pseudo-processes.

N.Smorodina, M.Faddeev, 2010 - generalized function theory.

Namely, the symmetric stable distribution with $\alpha > 2$ was defined as a generalized function I that acts on a test function φ as

$$(I, \varphi) = \lim_{\varepsilon \to 0} \mathbf{E} \varphi * \omega_\varepsilon(\eta_\varepsilon), \tag{7}$$

where ω_ε is a special family of rapidly oscillating functions, $\eta_\varepsilon = \int_{|x|>\varepsilon} x d\mu$, and μ is a Poisson random measure on $\mathbb{R}$ with intensity measure $\frac{C_\alpha dx}{|x|^{1+\alpha}}$. If $\alpha \in (0, 2)$, then in (7) for every ε the function ω_ε is $\delta-$function and in this case the generalized function I is a regular functional of the form

$$(I, \varphi) = \int_{-\infty}^{\infty} \varphi(x) p_\alpha(x) dx,$$

where $p_\alpha(x)$ is a density of the symmetric stable distribution with index α. For $\alpha > 2$ the generalized function I is a regular functional, but corresponding density is the function with alternating signs.

Note, that this method works well only if $\alpha \in \bigcup_{m=1}^{\infty} (4m, 4m+2)$, in this case the Fourier transform $g_\alpha(p)$ of the stable distribution (defined by (7)) has the "right" form (as for $\alpha \in (0,2)$), namely

$$g_\alpha(p) = \exp\left(-c\,|p|^\alpha\right),$$

where c is a positive constant.
For $\alpha \in \bigcup_{m=1}^{\infty} (4m-2, 4m)$ the method of Smorodina, Faddeev gives us not so "natural" result, namely

$$g_\alpha(p) = \exp\left(c_0\,|p|^\alpha - c_1 p^{4m}\right).$$

For $\alpha \in \bigcup_{m=1}^{\infty}(4m, 4m+2)$ we also used the methods of Smorodina, Faddeev only, but in the case $\alpha \in \bigcup_{m=1}^{\infty}(4m-2, 4m)$ we suggest a new method based on the theory of Hardy classes. In fact, instead of one real-valued process we consider two complex-valued processes (in the nonsymmetric case) and four complex-valued processes (in the symmetric case). Note that this method provides us the "right" view of the Fourier transform

$$g_\alpha(p) = \exp(-c(p)|p|^\alpha)$$

for any α, where c(p) depends on sign(p) in the nonsymmetric case and does not depend on p in the symmetric case.

Let $\nu(dx, dt)$ be a Poisson random measure on $\mathbb{R} \times [0, T]$ with intensity measure $\mathbf{E}\nu(dx, dt) = \Lambda(dx) \cdot dt = \frac{dx \cdot dt}{|x|^{1+\alpha}}$, $\alpha > 2$ and $\alpha \notin \mathbb{N}$.
Denote $\mathbb{R}_\varepsilon = \mathbb{R} \setminus (-\varepsilon, \varepsilon)$.
For $\varepsilon>0$ by $\xi_\varepsilon^+(t)$ we denote the random process

$$\xi_\varepsilon^+(t) = \iint\limits_{[0,t]\times(\varepsilon,+\infty)} x\nu(dx, dt). \tag{8}$$

and by $\xi_\varepsilon(t)$ we denote the random process

$$\xi_\varepsilon(t) = \iint\limits_{[0,t]\times\mathbb{R}_\varepsilon} x\nu(dx, dt). \tag{9}$$

The characteristic function of $\xi_\varepsilon^+(t)$ is

$$f_{\xi_\varepsilon^+(t)}(p) = \exp(t \int\limits_{\varepsilon}^{+\infty} (e^{ipx} - 1)\Lambda(dx)).$$

The characteristic function of $\xi_\varepsilon(t)$ is

$$f_{\xi_\varepsilon(t)}(p) = \exp(t \int\limits_{\mathbb{R}_\varepsilon} (e^{ipx} - 1)\Lambda(dx)).$$

For the case $\alpha \in \bigcup\limits_{m=1}^{\infty}(4m, 4m+1) \cup (4m+1, 4m+2)$ and $\alpha \in \bigcup\limits_{m=1}^{\infty}(4m-2, 4m-1) \cup (4m-1, 4m)$ we use different approaches. Consider the first case.

The case $\alpha \in \bigcup_{m=1}^{\infty} (4m, 4m+1) \cup (4m+1, 4m+2)$

Firstly, we consider the nonsymmetric case.
For $\varepsilon > 0$ define a function $u_\varepsilon(t,x)$ by

$$u_\varepsilon(t,x) = \mathbf{E}[(\varphi * \omega_\varepsilon^t)(x - \xi_\varepsilon^+(t))],$$

where

$$\widehat{\omega}_\varepsilon^t(p) = \exp\Big(- t \int_\varepsilon^{+\infty} \Big(\sum_{k=1}^{[\alpha]} \frac{i^k p^k x^k}{k!} \Big) \frac{dx}{x^{1+\alpha}} \Big). \qquad (10)$$

Theorem

Suppose that $\varphi \in W_2^{l+[\alpha]+1}(\mathbb{R})$, $l \geqslant 0$ *and let* $u(t,x)$ *be a solution of the Cauchy problem* (1), (4). *Then there exists* $C = C(\alpha) > 0$, *such that*

$$\sup_{t \in [0,T]} \|u_\varepsilon(t,\cdot) - u(t,\cdot)\|_{W_2^l(\mathbb{R})} \leqslant CT \, \|\varphi\|_{W_2^{l+[\alpha]+1}(\mathbb{R})} \, \varepsilon^{1-\{\alpha\}}.$$

The case $\alpha \in \bigcup_{m=1}^{\infty} (4m, 4m+1) \cup (4m+1, 4m+2)$

Now we consider the symmetric case. For $\varepsilon > 0$ define a function $u_\varepsilon(t,x)$ by

$$u_\varepsilon(t,x) = \mathbf{E}[(\varphi * \omega_\varepsilon^t)(x - \xi_\varepsilon(t))],$$

where

$$\widehat{\omega}_\varepsilon^t(p) = \exp\Big(- t \int\limits_{\mathbb{R}_\varepsilon} \Big(\sum_{k=1}^{2m} \frac{i^{2k} p^{2k} x^{2k}}{(2k)!} \Big) \frac{dx}{|x|^{1+\alpha}} \Big). \qquad (11)$$

Theorem

Suppose that $\varphi \in W_2^{l+4m+2}(\mathbb{R})$, $l \geqslant 0$ and let $u(t,x)$ be a solution of the Cauchy problem (3), (4). Then there exists $C = C(\alpha) > 0$, such that

$$\sup_{t \in [0,T]} \|u_\varepsilon(t,\cdot) - u(t,\cdot)\|_{W_2^l(\mathbb{R})} \leqslant CT \, \|\varphi\|_{W_2^{l+4m+2}(\mathbb{R})} \, \varepsilon^{4m+2-\alpha}.$$

For $M > 0$ by P_M denote the projector in $L_2(\mathbb{R})$ on the subspace of the functions ψ, such that $\operatorname{supp}\widehat{\psi} \subset [-M, M]$. Namely, for $\psi \in L_2(\mathbb{R})$ set

$$\mathrm{P}_M\psi = \psi * D_M,$$

where D_M is the Dirichlet kernel

$$D_M(x) = \frac{1}{\pi}\frac{\sin Mx}{x}.$$

The Fourier transform $\widehat{\mathrm{P}}_M$ of the operator P_M is a multiplication operator of the form

$$\widehat{\mathrm{P}}_M\widehat{\psi} = \widehat{\psi} \cdot \widehat{D}_M,$$

where $\widehat{D}_M(p) = \mathbf{1}_{[-M,M]}(p)$.
We use the notation $\psi_M(x)$ for $P_M\psi(x)$.

The case $\alpha \in \bigcup_{m=1}^{\infty} (4m-2, 4m-1) \cup (4m-1, 4m)$

Denote $\mathbb{R}_\varepsilon^+ = (\varepsilon, +\infty)$ and $\mathbb{R}_\varepsilon^- = (-\infty, -\varepsilon)$.
As above for $\varepsilon > 0$ we define random processes $\xi_\varepsilon^\pm(t)$ by
$\xi_\varepsilon^\pm(t) = \iint\limits_{[0,t]\times\mathbb{R}_\varepsilon^\pm} x\nu(dx, dt)$. But now we consider the complex-valued processes $\sigma\xi_\varepsilon^\pm(t)$, where σ is a complex constant.
For $\xi_\varepsilon^+(t)$ we have

$$\mathbf{E}\exp(ip\sigma\xi_\varepsilon^+(t)) = \exp(t\int\limits_\varepsilon^{+\infty}(e^{i\sigma px} - 1)\Lambda(dx)).$$

This integral converges if $p \geqslant 0$ and $\operatorname{Im}\sigma \geqslant 0$ or if $p \leqslant 0$ and $\operatorname{Im}\sigma \leqslant 0$.

For $\xi_\varepsilon^-(t)$ we have

$$\mathbf{E}\exp(ip\sigma\xi_\varepsilon^-(t)) = \exp(t\int\limits_{-\infty}^{-\varepsilon}(e^{i\sigma px}-1)\Lambda(dx)).$$

This integral converges if $p \geqslant 0$ and $\mathrm{Im}\,\sigma \leqslant 0$ or if $p \leqslant 0$ and $\mathrm{Im}\,\sigma \geqslant 0$.
By P_+ we denote the Riesz projector. This projector acts from $L_2(\mathbb{R})$ to Hardy space $H^2_+(\{\mathrm{Im} z > 0\})$. Analogously, the projector P_- acts from $L_2(\mathbb{R})$ to Hardy space $H^2_-(\{\mathrm{Im} z < 0\})$. So that for every $\varphi \in L_2(\mathbb{R})$ we have

$$\varphi = \varphi_+ + \varphi_- = P_+\varphi + P_-\varphi.$$

Set $\sigma_+ = \exp(\frac{i\pi}{\alpha})$ and $\sigma_- = \exp(-\frac{i\pi}{\alpha})$. Note that σ_+ belongs to the upper half-plane and σ_- belongs to the lower half-plane and

$$\sigma_\pm^\alpha = -1.$$

Now we consider the nonsymmetric case.
For $\varepsilon>0$ define a function $u_\varepsilon(t,x)$ by

$$u_\varepsilon(t,x) = \mathbf{E}[(\varphi_M^- * \omega_\varepsilon^t)(x - \sigma_+\xi_\varepsilon^+(t)) + (\varphi_M^+ * \omega_\varepsilon^t)(x - \sigma_-\xi_\varepsilon^+(t))],$$

where

$$\widehat{\omega}_\varepsilon^t(p) = \begin{cases} \exp\left(-t\int\limits_\varepsilon^{+\infty}\left(\sum\limits_{k=1}^{[\alpha]}\frac{i^k\sigma_+^k p^k x^k}{k!}\right)\frac{dx}{x^{1+\alpha}}\right), & p \geqslant 0, \\ \exp\left(-t\int\limits_\varepsilon^{+\infty}\left(\sum\limits_{k=1}^{[\alpha]}\frac{i^k\sigma_-^k p^k x^k}{k!}\right)\frac{dx}{x^{1+\alpha}}\right), & p < 0. \end{cases} \quad (12)$$

Theorem

Suppose that $\varphi \in W_2^{l+[\alpha]+1}(\mathbb{R})$, $l \geqslant 0$ *and* $M(\varepsilon) = \frac{1}{\varepsilon}$. *Let* $u(t,x)$ *be a solution of the Cauchy problem* (1), (4). *Then there exists* $C = C(\alpha) > 0$, *such that*

$$\sup_{t\in[0,T]} \|u_\varepsilon(t,\cdot) - u(t,\cdot)\|_{W_2^l(\mathbb{R})} \leqslant C(T + \varepsilon^\alpha)\|\varphi\|_{W_2^{l+[\alpha]+1}(\mathbb{R})}\,\varepsilon^{1-\{\alpha\}}.$$

The case $\alpha \in \bigcup_{m=1}^{\infty} (4m-2, 4m-1) \cup (4m-1, 4m)$

In the symmetric case for $\varepsilon > 0$ we define a function $u_\varepsilon(t,x)$ by

$$u_\varepsilon(t,x) = \mathbf{E}[(\varphi_M^- * \omega_\varepsilon^t)(x - \sigma_+\xi_\varepsilon^+(t) - \sigma_-\xi_\varepsilon^-(t)) \\ + (\varphi_M^+ * \omega_\varepsilon^t)(x - \sigma_-\xi_\varepsilon^+(t) - \sigma_+\xi_\varepsilon^-(t))],$$

where

$$\widehat{\omega}_\varepsilon^t(p) \\ = \begin{cases} \exp\left(-t\int\limits_{\varepsilon}^{+\infty}\left(\sum\limits_{k=1}^{[\alpha]}\frac{(i\sigma_+px)^k}{k!}\right)\frac{dx}{x^{1+\alpha}}\right)\exp\left(-t\int\limits_{-\infty}^{-\varepsilon}\left(\sum\limits_{k=1}^{[\alpha]}\frac{(i\sigma_-px)^k}{k!}\right)\frac{dx}{|x|^{1+\alpha}}\right), \\ \qquad \text{if } p \geqslant 0, \\ \exp\left(-t\int\limits_{\varepsilon}^{+\infty}\left(\sum\limits_{k=1}^{[\alpha]}\frac{(i\sigma_-px)^k}{k!}\right)\frac{dx}{x^{1+\alpha}}\right)\exp\left(-t\int\limits_{-\infty}^{-\varepsilon}\left(\sum\limits_{k=1}^{[\alpha]}\frac{(i\sigma_+px)^k}{k!}\right)\frac{dx}{|x|^{1+\alpha}}\right), \\ \qquad \text{if } p < 0. \end{cases}$$

Theorem

Suppose that $\varphi \in W_2^{l+[\alpha]+1}(\mathbb{R})$, $l \geqslant 0$ *and* $M(\varepsilon) = \varepsilon^{-1}$. *Let* $u\,(t,x)$ *be a solution of the Cauchy problem* (3), (4). *Then there exists* $C = C\,(\alpha) > 0$, *such that*

$$\sup_{t\in[0,T]} \|u_\varepsilon(t,\cdot) - u(t,\cdot)\|_{W_2^l(\mathbb{R})} \leqslant C(T+\varepsilon^\alpha)\,\|\varphi\|_{W_2^{l+[\alpha]+1}(\mathbb{R})}\,\varepsilon^{1-\{\alpha\}}.$$

Thus we get a probability representation of the Cauchy problem solution (1), (4) defined by

$$u(t,x) = \lim_{\varepsilon\to 0} \mathbf{E}[(\varphi_M^- * \omega_\varepsilon^t)(x - \sigma_+\xi_\varepsilon^+(t)) + (\varphi_M^+ * \omega_\varepsilon^t)(x - \sigma_-\xi_\varepsilon^+(t))]$$

and a probability representation of the Cauchy problem solution (3), (4) defined by

$$u(t,x) = \lim_{\varepsilon\to 0} \mathbf{E}\Big[(\varphi_M^+ * \omega_\varepsilon^t)(x - \sigma_-\xi_\varepsilon^+(t) - \sigma_+\xi_\varepsilon^-(t)) + (\varphi_M^- * \omega_\varepsilon^t)(x - \sigma_+\xi_\varepsilon^+(t) - \sigma_-\xi_\varepsilon^-(t))\Big].$$

Limit theorems

Let $\left\{\xi_j^+\right\}_{j=1}^{\infty}$ be a sequence of i.i.d. nonnegative random variables and $\left\{\xi_j^-\right\}_{j=1}^{\infty}$ be a sequence of i.i.d. nonpositive random variables.
Suppose that the distributions $\mathcal{P}^{\pm}$ of $\xi_1^{\pm}$ for $|x| > 1$ satisfy the conditions

$$P(\xi_1^{\pm} > |x|) = \frac{1}{\alpha |x|^{\alpha}}(1 + h^{\pm}(x)), \qquad (13)$$

where $|h^{\pm}(x)| \leqslant \frac{C}{|x|^{\beta}}$, and $\beta > 1 - \{\alpha\}$.
For $k < \alpha$ by $\mu_k^{\pm} = \mathbf{E}(\xi_1^{\pm})^k$ we denote the moment of the order k.
Let $\eta(t)$, $t \in [0, \infty)$ be a standard Poisson process independent of $\{\xi_j^{\pm}\}$.
Define random processes $\zeta_n^{\pm}(t)$, $t \in [0, T]$, by

$$\zeta_n^{\pm}(t) = \frac{1}{n^{1/\alpha}} \sum_{j=1}^{\eta(nt)} \xi_j^{\pm}. \qquad (14)$$

In the nonsymmetric case for $n \in \mathbb{N}$ define a function

$$u_n(t,x) = \mathbf{E}[(\varphi_M * \varkappa_n^t)(x - \zeta_n^+(t))],$$

where

$$\widehat{\varkappa}_n^t(p) = \exp(-nt(\frac{\mu_1^+ ip}{n^{1/\alpha}} + \frac{\mu_2^+(ip)^2}{2n^{2/\alpha}} + \cdots + \frac{\mu_{[\alpha]}^+(ip)^{[\alpha]}}{[\alpha]!n^{[\alpha]/\alpha}})).$$

We choose $M = M(n)$.

Theorem

Suppose that $\varphi \in W_2^{l+[\alpha]+1}(\mathbb{R})$, $l \geqslant 0$, $M(n) = n^{1/\alpha}$ *and let* $u(t,x)$ *be a solution of the Cauchy problem* (1), (4). *Then there exists* $C = C(\alpha) > 0$, *such that*

$$\sup_{t\in[0,T]} \|u_n(t,\cdot) - u(t,\cdot)\|_{W_2^l(\mathbb{R})} \leqslant C\left(T + \frac{1}{n}\right)\frac{\|\varphi\|_{W_2^{l+[\alpha]+1}(\mathbb{R})}}{n^{(1-\{\alpha\})/\alpha}}.$$

The case $\alpha \in \bigcup_{m=1}^{\infty} (4m, 4m+1) \cup (4m+1, 4m+2)$

Now we consider the symmetric case. Let $\{\xi_j\}_{j=1}^{\infty}$ be a sequence of i.i.d. symmetric random variables. Suppose that the distribution $\mathcal{P}$ of ξ_1 for $x > 1$ satisfies the condition

$$P(\xi_1 > x) = \frac{1}{\alpha |x|^{\alpha}} (1 + h(x)), \qquad (15)$$

where $|h(x)| \leqslant \frac{C}{|x|^{\beta}}$, and $\beta > 4m + 2 - \alpha$.

For $k < \alpha$ by $\mu_k = \mathbf{E}\xi_1^k$ we denote the moment of the order k.

Let $\eta(t)$, $t \in [0, \infty)$ be a standard Poisson process independent of $\{\xi_j\}$.

Define a random process $\zeta_n(t)$, $t \in [0, T]$, by

$$\zeta_n(t) = \frac{1}{n^{1/\alpha}} \sum_{j=1}^{\eta(nt)} \xi_j. \qquad (16)$$

For $n \in \mathbb{N}$ define a function

$$u_n(t,x) = \mathbf{E}[(\varphi_M * \varkappa_n^t)(x - \zeta_n(t))],$$

where

$$\widehat{\varkappa}_n^t(p) = \exp\Big(-nt\Big(\frac{\mu_2 (ip)^2}{2n^{2/\alpha}} + \cdots + \frac{\mu_{4m}(ip)^{4m}}{(4m)!\,n^{4m/\alpha}}\Big)\Big).$$

We choose $M = M(n)$.

Theorem

Suppose that $\varphi \in W_2^{l+4m+2}(\mathbb{R})$, $l \geqslant 0$, $M(n) = n^{1/\alpha}$ and let $u(t,x)$ be a solution of the Cauchy problem (3), (4). Then there exists $C = C(\alpha) > 0$, such that

$$\sup_{t\in[0,T]} \|u_n(t,\cdot) - u(t,\cdot)\|_{W_2^l(\mathbb{R})} \leqslant C\Big(T + \frac{1}{n}\Big)\frac{\|\varphi\|_{W_2^{l+4m+2}(\mathbb{R})}}{n^{(4m+2-\alpha)/\alpha}}.$$

The case $\alpha \in \bigcup_{m=1}^{\infty}(4m-2,4m-1)\cup(4m-1,4m)$

In the nonsymmetric case for $n \in \mathbb{N}$ define a function $u_n(t,x)$

$$u_n(t,x) = \mathbf{E}[(\varphi_M^- * \varkappa_n^t)(x - \sigma_+\zeta_n^+(t)) + (\varphi_M^+ * \varkappa_n^t)(x - \sigma_-\zeta_n^+(t))],$$

where

$$\widehat{\varkappa}_n^t(p) = \begin{cases} \exp\left(-nt\left(\frac{\mu_1^+ ip\sigma_+}{n^{1/\alpha}} + \cdots + \frac{\mu_{[\alpha]}^+ (ip\sigma_+)^{[\alpha]}}{[\alpha]!n^{[\alpha]/\alpha}}\right)\right), & p \geqslant 0, \\ \exp\left(-nt\left(\frac{\mu_1^+ ip\sigma_-}{n^{1/\alpha}} + \cdots + \frac{\mu_{[\alpha]}^+ (ip\sigma_-)^{[\alpha]}}{[\alpha]!n^{[\alpha]/\alpha}}\right)\right), & p < 0. \end{cases}$$

Theorem

Suppose that $\varphi \in W_2^{l+[\alpha]+1}(\mathbb{R})$, $l \geqslant 0$, $M(n) = n^{1/\alpha}$ *and let* $u(t,x)$ *be a solution of the Cauchy problem* (1), (4). *Then there exists* $C = C(\alpha) > 0$, *such that*

$$\sup_{t\in[0,T]} \|u_n(t,\cdot) - u(t,\cdot)\|_{W_2^l(\mathbb{R})} \leqslant C(T + \frac{1}{n})\frac{\|\varphi\|_{W_2^{l+[\alpha]+1}(\mathbb{R})}}{n^{(1-\{\alpha\})/\alpha}}.$$

The case $\alpha \in \bigcup_{m=1}^{\infty} (4m-2, 4m-1) \cup (4m-1, 4m)$

In the symmetric case for $n \in \mathbb{N}$ define a function $u_n(t,x)$

$$u_n(t,x) = \mathbf{E}\big[\big(\varphi_M^- * \varkappa_n^t\big)\left(x - \sigma_+\zeta_n^+(t) - \sigma_-\zeta_n^-(t)\right) + \big(\varphi_M^+ * \varkappa_n^t\big)\left(x - \sigma_-\zeta_n^+(t) - \sigma_+\zeta_n^-(t)\right)\big],$$

where

$$\widehat{\varkappa}_n^t(p) = \begin{cases} \exp\left(-nt\left(\sum_{k=1}^{[\alpha]} \frac{\mu_k^+(i\sigma_+ p)^k}{k!}\right)\right)\exp\left(-nt\left(\sum_{k=1}^{[\alpha]} \frac{\mu_k^-(i\sigma_- p)^k}{k!}\right)\right), & \text{if } p \geqslant 0, \\ \exp\left(-nt\left(\sum_{k=1}^{[\alpha]} \frac{\mu_k^+(i\sigma_- p)^k}{k!}\right)\right)\exp\left(-nt\left(\sum_{k=1}^{[\alpha]} \frac{\mu_k^-(i\sigma_+ p)^k}{k!}\right)\right), & \text{if } p < 0. \end{cases}$$

Theorem

Suppose that $\varphi \in W_2^{l+[\alpha]+1}(\mathbb{R})$, $l \geqslant 0$, $M(n) = n^{1/\alpha}$ *and let* $u(t,x)$ *be a solution of the Cauchy problem* (3), (4). *Then there exists* $C = C(\alpha) > 0$, *such that*

$$\sup_{t\in[0,T]} \|u_n(t,\cdot) - u(t,\cdot)\|_{W_2^l(\mathbb{R})} \leqslant C\big(T + \frac{1}{n}\big)\frac{\|\varphi\|_{W_2^{l+[\alpha]+1}(\mathbb{R})}}{n^{(1-\{\alpha\})/\alpha}}.$$

References

M.V. Platonova: Symmetric α-stable distributions for noninteger $\alpha > 2$ and associated stochastic processes. Zapiski Nauchnykh Seminarov POMI 442, 101-117, St. Petersburg Department of Steklov Institute of Mathematics, Russian Academy of Sciences, 2015 (in Russian).

A. Lachal: From pseudo-random walk to pseudo-Brownian motion: first exit time from a one-sided or a two-sided interval. ArXiv:1301.6579.

E. Orsingher, B. Toaldo: Pseudoprocesses related to space-fractional higher-order heat-type equations. Stochast. Anal. Appl. **32**, No. 4 (2014), 619–641.

Smorodina N.V., Faddeev M.M.: The Lévy-Khinchin representation of the one class of signed stable measures and some its applications. Acta Appl. Math. 110, 1289-1308, 2010.