Infinitesimality of operators with non-basis family of eigenvectors

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The fundamental concept for the study of linear evolution equations

$$\begin{cases} \dot{x}(t) = Ax(t), & t \geq 0, \\ x(0) = x_0 \in X, \end{cases} \tag{1}$$

is the concept of C_0-semigroup.

A one-parameter family $\{T(t)\}_{t\geq 0} : \mathbb{R}_+ \mapsto [X]$ – C_0-semigroup if:

1. $T(t)T(s) = T(t+s),\ t, s \geq 0;$
2. $T(0) = I;$
3. $\lim\limits_{t\downarrow 0} \|T(t)x - x\| = 0,\ x \in X.$

C_0-semigroups play important role in operator theory, theory of PDE's and infinite-dimensional linear systems theory.

Generator of C_0-semigroup $\{T(t)\}_{t\geq 0}$ – operator $A : X \supset D(A) \mapsto X$,

which acts by the formula $Ax = \lim\limits_{t\downarrow 0} \frac{T(t)x - x}{t}$, $x \in D(A)$, with

$$D(A) = \left\{ x \in X : \ \exists \lim_{t\downarrow 0} \frac{T(t)x - x}{t} \right\}.$$

The operator A is an infinitesimal generator (generator)

of C_0-semigroup in $X \Leftrightarrow$ the Cauchy problem (1) is well-posed and $\rho(A) \neq \emptyset$. The solution is given by

$$x(\cdot, x_0) = T(\cdot)x_0.$$

$x_0 \in D(A) \implies$ classical solution
$x_0 \in X \implies$ mild solution

Example (This phenomenon takes place for:)

1. Maxwell's equations of electrodynamics
2. Systems of differential equations with delay
3. Regular Sturm-Liouville systems
4. Linear heat and wave equations

Central problems of C_0-semigroup theory are

1. To examine whether a concrete operator A is the generator of C_0-semigroup (to examine the infinitesimality of A), and
2. To obtain the representation of this C_0-semigroup.

The criterion of infinitesimality of A:

Theorem (E. Hille, K. Yosida, R. Phillips, W. Feller, I. Miyadera)

The operator $A : X \supset D(A) \mapsto X$ is the infinitesimal generator of C_0-semigroup $\{T(t)\}_{t \geq 0}$ satisfying $\|T(t)\| \leq Me^{\omega t}$ if and only if

1. $D(A)$ is dense, A is closed, and
2. $(\omega, +\infty) \subseteq \rho(A)$ and $\forall \lambda > \omega$, $\forall n \in \mathbb{N}$ we have

$$\left\|(\lambda I - A)^{-n}\right\| \leq \frac{M}{(\lambda - \omega)^n}.$$

But this theorem can be extremely rare used in practice because of complexity of conditions 1 and 2. The Lumer-Phillips theorem is much more useful but it covers only the case of contraction semigroups ($M = 1$, $\omega = 0$).

The Riesz-basis property is valuable in an infinite-dimensional linear systems theory.

This property is essentially used in the study of

1. Stability
2. Controllability
3. Stabilization
4. Observability
5. Spectral assignment
6. Asymptotic properties

of various infinite-dimensional linear systems.

In particular, R. Rabah, G. M. Sklyar, A. V. Rezounenko,

K. V. Sklyar, P. Barkhaev, P. Polak (University of Szczecin, Poland & V. N. Karazin Kharkiv University, Ukraine, 2003-2016) studied all these properties for linear delay systems of neutral type.

Theorem (G.Q. Xu & S.P. Yung, JDE, 2005, H. Zwart, JDE, 2010)

Let A be the generator of the C_0-group in H, with simple eigenvalues $\{\lambda_n\}_1^\infty$ and the corresp. (normalized) eigenvectors $\{\phi_n\}_1^\infty$. If $\overline{Lin}\{\phi_n\}_1^\infty = H$ and

$$\inf_{n \neq m} |\lambda_n - \lambda_m| > 0, \tag{2}$$

then $\{\phi_n\}_1^\infty$ forms a Riesz basis of H.

Theorem (H. Zwart, JDE, 2010)

Let A be the generator of the C_0-group in H with eigenvalues $\{\lambda_n\}_1^\infty$. If the system of generalized eigenvectors is dense and

$$\{\lambda_n\}_1^\infty = \bigcup_{j=1}^{K} \{\lambda_{n,j}\}_{n=1}^\infty, \text{ where } \inf_{n \neq m} |\lambda_{n,k} - \lambda_{m,k}| > 0, \; k = 1, \ldots, K, \tag{3}$$

then $\exists$ spectral projections $\{P_n\}_1^\infty$ of A such that $\{P_n H\}_1^\infty$ is a Riesz basis of subspaces of H and $\max_n \dim P_n H \leq K$.

What happens when eigenvalues do not satisfy the condition (3)?

In particular:

1. Is it possible to construct the generator A of the C_0-group with purely imaginary eigenvalues, which don't satisfy (3), and dense family of eigenvectors, which don't form a Schauder basis?
2. When the Cauchy problem with such an operator A is well/ill–posed?

In a joint work with Dr. Grigory Sklyar we obtain the following

Answers:

1. Yes, and we construct the class of generators of C_0-groups with these preassigned properties.
2. The well-posedness of the Cauchy problem with such an operator A essentially depends on the asymptotic behaviour of its eigenvalues $\{\lambda_n\}_1^\infty$ at $i\infty$. We found conditions on the asymptotic behaviour of $\{\lambda_n\}_1^\infty$ under which the corresponding Cauchy problem is well/ill–posed.

To obtain these results we

- Introduce and study special classes of Hilbert spaces $H_k(\{e_n\})$, $k \in \mathbb{N}$. Space $H_k(\{e_n\})$ depend on an arbitrary separable Hilbert space H and a chosen Riesz basis $\{e_n\}_1^\infty$ of H.
- Prove that $\{e_n\}_1^\infty$ is dense and minimal in $H_k(\{e_n\})$ but not uniformly minimal, hence do not form a Schauder basis.
- Consider the classes $\mathcal{S}_k$, $k \in \mathbb{N}$, of increasing sequences $\{f(n)\}_{n=1}^\infty \subset \mathbb{R}$ satisfying

  $$\left\{n^j \Delta^j f(n)\right\}_{n=1}^\infty \in \ell_\infty$$

  for $1 \le j \le k$, where Δ is a difference operator.

Example (For every $k \in \mathbb{N}$:)

1. $\{\ln n\}_{n=1}^\infty \in \mathcal{S}_k$, $\{\ln \ln(n+1)\}_{n=1}^\infty \in \mathcal{S}_k$,
2. $\left\{\ln \ln \sqrt{n+1}\right\}_{n=1}^\infty \in \mathcal{S}_k$,
3. $\left\{\sqrt{n}\right\}_{n=1}^\infty \notin \mathcal{S}_k$.

Spaces $H_k(\{e_n\})$, $k \in \mathbb{N}$

Choose separable Hilbert space H and let $\{e_n\}_1^\infty$ be an arbitrary Riesz basis in H. Then we define a Hilbert space $H_k(\{e_n\})$, $k \in \mathbb{N}$, as

$$H_k(\{e_n\}) = \left\{ x = (\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n : \ \{c_n\}_1^\infty \in \ell_2(\Delta^k) \right\}, \ k \in \mathbb{N},$$

with $\left\| (\mathfrak{f}) \sum\limits_{n=1}^{\infty} c_n e_n \right\|_k = \left\| \sum\limits_{n=1}^{\infty} \left(\Delta^k c_n\right) e_n \right\| = \left\| \sum\limits_{n=1}^{\infty} \sum\limits_{j=0}^{k} (-1)^j C_k^j c_{n-j} e_n \right\|.$

Here $\ell_2(\Delta^k) = \{\alpha = \{\alpha_n\}_{n=1}^\infty : \ \Delta^k \alpha \in \ell_2\}$.

The space $\ell_2(\Delta)$ was first introduced and studied by F. Başar & B. Altay, Ukrainian Math. J., 2003. Later, in 2006, the space $\ell_2(\Delta^k)$, $k \in \mathbb{N}$, was studied by B. Altay, Studia Sci. Math. Hungar.

$H_k(\{e_n\})$, $k \in \mathbb{N}$, is isomorphic to ℓ_2 and the following holds:

$H \subset H_1(\{e_n\}) \subset H_2(\{e_n\}) \subset H_3(\{e_n\}) \subset \dots$

In particular case when $k = 1$ and $\{e_n\}_{n=1}^{\infty}$ is a canonical basis of ℓ_2,

$e_1 = (1, 0, 0, 0, 0, \dots)^T$, $e_2 = (0, 1, 0, 0, 0, \dots)^T$, $e_3 = (0, 0, 1, 0, 0, \dots)^T$, $e_4 = (0, 0, 0, 1, 0, \dots)^T, \dots$ we have

$$H_1(\{e_n\}) = \ell_2(\Delta) = \left\{ \{c_n\}_{n=1}^{\infty} \subset \mathbb{C} : \sum_{n=1}^{\infty} |c_n - c_{n-1}|^2 < \infty,\ c_0 = 0 \right\}.$$

Then we have that

- For any $\alpha \in \left[0, \frac{1}{2}\right)$ we have $(1, 2^\alpha, 3^\alpha, 4^\alpha, 5^\alpha, \dots)^T \in \ell_2(\Delta)$; Indeed, for $\alpha = 0$ this is obvious. If $\alpha \in (0, \frac{1}{2})$, then

$$n^\alpha - (n-1)^\alpha \sim c_\alpha n^{\alpha - 1},\ n \to \infty,$$

where c_α – constant depending on α. Consequently $\{n^\alpha - (n-1)^\alpha\}_{n=1}^{\infty} \in \ell_2 \implies (1, 2^\alpha, 3^\alpha, 4^\alpha, 5^\alpha, \dots)^T \in \ell_2(\Delta)$.

- $\overline{Lin}\{e_n\}_{n=1}^{\infty} = \ell_2(\Delta)$, because only zero is orthogonal to all e_n, $n \in \mathbb{N}$.
- $\{e_n\}_{n=1}^{\infty}$ do not form a Schauder basis of $\ell_2(\Delta)$;

Suppose the opposite, i.e. that $\{e_n\}_{n=1}^{\infty}$ is a basis of $\ell_2(\Delta)$. Then for every $x \in \ell_2(\Delta)$ we have $x = \sum_{n=1}^{\infty} c_n e_n$. Since $\|e_n\|_1 = \sqrt{2}$ for each n, then, by the necessary condition of convergence of series, we will have $c_n \to 0$, for $n \to \infty$. Consider $x = (\mathfrak{f}) \sum_{n=1}^{\infty} e_n = (1, 1, 1, 1, \ldots)^T \in \ell_2(\Delta)$. Then we arrive at

$$(1, 1, 1, 1, \ldots)^T = (c_1, c_2, c_3, c_4, \ldots)^T,$$

where $c_n \to 0$, for $n \to \infty$ – a contradiction.

Theorem (Central construction)

An operator $A : \ell_2(\Delta) \supset D(A) \mapsto \ell_2(\Delta)$, defined by

$$A \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ \dots \end{pmatrix} = \begin{pmatrix} 0 \\ i \ln 2 \cdot c_2 \\ i \ln 3 \cdot c_3 \\ i \ln 4 \cdot c_4 \\ \dots \end{pmatrix},$$

with domain

$$D(A) = \{\{c_n\}_{n=1}^{\infty} \in \ell_2(\Delta) : \ \{\ln n \cdot c_n\}_{n=1}^{\infty} \in \ell_2(\Delta)\},$$

generates the C_0-group $\{e^{At}\}_{t\in\mathbb{R}}$ on $\ell_2(\Delta)$, which is given by the formula

$$e^{At} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ \dots \end{pmatrix} = \begin{pmatrix} c_1 \\ e^{it\ln 2} c_2 \\ e^{it\ln 3} c_3 \\ e^{it\ln 4} c_4 \\ \dots \end{pmatrix}, \ t \in \mathbb{R}. \tag{4}$$

Example (Operator with non-basis family of eigenvectors)

- Consider operator $\mathcal{L} : D(\mathcal{L}) \mapsto L_2(\mathbb{R}, \mathbb{C})$, defined by

$$\mathcal{L}\psi = -\psi'' + ix\psi, \quad \psi \in D(\mathcal{L}),$$

$D(\mathcal{L}) = \left\{\psi \in L_2(\mathbb{R}_+, \mathbb{C}) : \; x\psi \in L_2(\mathbb{R}_+, \mathbb{C}), \psi \in H_0^2(\mathbb{R}_+, \mathbb{C})\right\}$. Let $\{\mu_n\}_{n=1}^{\infty} \subset \mathbb{R}$ – decreasing sequence of zeros of Airy function $Ai(z)$. Then $\{\lambda_n\}_{n=1}^{\infty}$, where $\lambda_n = e^{-\frac{2\pi i}{3}}\mu_n, \quad n \in \mathbb{N}$, includes all eigenvalues of $\mathcal{L}$. Since $\lim\limits_{n\to\infty} \mu_n = -\infty$ and $\lim\limits_{n\to\infty} |\mu_{n+1} - \mu_n| = 0$, then $\{\lambda_n\}_{n=1}^{\infty}$ satisfy $\lim\limits_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$. Eigenfunctions of $\mathcal{L}$ are

$$\tilde{u}_n = Ai\left(e^{\frac{\pi i}{6}} x + \mu_n\right) \in H_0^2(\mathbb{R}_+, \mathbb{C}), \quad n \in \mathbb{N}.$$

Normalized eigenfunctions $u_n = \frac{\tilde{u}_n}{\|\tilde{u}_n\|}$, $n \in \mathbb{N}$, are complete in $L_2(\mathbb{R}, \mathbb{C})$, but don't form a Schauder basis in $L_2(\mathbb{R}, \mathbb{C})$, see

Y. Almog, The stability of the normal state of superconductors in the presence of electric currents, SIAM J. Math. Anal., 2008, Vol. 40, pp. 824–850. (Ginzburg-Landau model)

Proposition (Spaces $H_k(\{e_n\})$, $k \in \mathbb{N}$, have the following properties:)

1. $\overline{Lin}\{e_n\}_{n=1}^{\infty} = H_k(\{e_n\})$;
2. $\{e_n\}_{n=1}^{\infty}$ does not form a basis of $H_k(\{e_n\})$;
3. $\{e_n\}_{n=1}^{\infty}$ has a unique biorthogonal system
$$\left\{\chi_n = (I-T)^{-k}(I-T^*)^{-k} e_n^*\right\}_{n=1}^{\infty}$$
in $H_k(\{e_n\})$, where $Te_n = e_{n+1}$, $n \in \mathbb{N}$, and $\langle e_n, e_m^* \rangle = \delta_n^m$;
4. $\{\chi_n\}_{n=1}^{\infty}$ is uniformly minimal sequence in $H_k(\{e_n\})$, $\{e_n\}_{n=1}^{\infty}$ is minimal but not uniformly minimal in $H_k(\{e_n\})$;
5. $H_k(\{e_n\})$ is Hilbert space, isomorphic to ℓ_2;
6. $L = \left\{x = (\mathfrak{f})\sum_{n=1}^{\infty} c_n e_n \in H_k(\{e_n\}) : \{c_n\}_{n=1}^{\infty} \in \ell_2(\Delta^k) \cap c_0\right\}$, where c_0 is the space of sequences $\{\alpha_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} \alpha_n = 0$, is not a (closed) subspace of $H_k(\{e_n\})$.

Theorem (The generalization)

Let $k \in \mathbb{N}$. Then the operator $A_k : H_k(\{e_n\}) \supset D(A_k) \mapsto H_k(\{e_n\})$, defined by

$$A_k x = A_k(\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n = (\mathfrak{f}) \sum_{n=1}^{\infty} if(n) \cdot c_n e_n,$$

where $\{f(n)\}_{n=1}^{\infty} \in \mathcal{S}_k = \left\{ \{f(n)\}_1^{\infty} : \lim_{n\to\infty} f(n) = +\infty; \right.$
$\left. \{n^j \Delta^j f(n)\}_{n=1}^{\infty} \in \ell_\infty \text{ for } 1 \leq j \leq k \right\}$, with domain

$$D(A_k) = \left\{ x = (\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n \in H_k(\{e_n\}) : \{f(n) \cdot c_n\}_{n=1}^{\infty} \in \ell_2(\Delta^k) \right\},$$

generates the C_0-group $\{e^{A_k t}\}_{t\in\mathbb{R}}$ on $H_k(\{e_n\})$, which is given by

$$e^{A_k t} x = e^{A_k t}(\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n = (\mathfrak{f}) \sum_{n=1}^{\infty} e^{itf(n)} c_n e_n, \ t \in \mathbb{R}. \tag{5}$$

Multiple application of the discrete Hardy inequality for $p = 2$

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^2 \leq 4\sum_{n=1}^{\infty} a_n^2$$

plays the key role in the proof of these theorems.

In the proof we also use the Leibnitz theorem for finite differences,

$$\Delta^k(u_n v_n) = \sum_{j=0}^{k} C_k^j \Delta^{k-j} u_{n-j} \Delta^j v_n, \quad k \in \mathbb{N},$$

and the following formula,

$$\Delta^d c_n = \sum_{m=1}^{n} \Delta^{d+1} c_m, \quad d, n \in \mathbb{N}.$$

Let $m : 1 \leq m \leq k$. Consider the following sets $\Sigma_1 = \{0, 1, 2, \ldots, k-1\}$, $\Sigma_2 = \{0, 1, 2, \ldots, k-2\}, \ldots, \Sigma_{k-1} = \{0, 1\}$, $\Sigma_k = \{0\}$. Clearly, $\Sigma_1 \supset \Sigma_2 \supset \Sigma_3 \supset \cdots \supset \Sigma_k$. One of the essential ingredients of the proof of this theorem is the following fact.

Proposition

For every $m : 1 \leq m \leq k$, each $\left\{\widetilde{f}(n)\right\}_{n=1}^{\infty} \in \mathcal{S}_k$, for all $s \in \Sigma_m$, $t \in \mathbb{R}$ and arbitrary $n > m$ the following inequality holds:

$$\left|\Delta^m e^{(-1)^s it\Delta^s \widetilde{f}(n)}\right| \leq \frac{\mathcal{P}_m\left[\widetilde{f}(n)\right](|t|)}{n^{m+s}}, \tag{6}$$

where $\mathcal{P}_m\left[\widetilde{f}(n)\right]$ is a polynomial of degree m, with positive coefficients depending on $\left\{\widetilde{f}(n)\right\}_{n=1}^{\infty}$, and without a free term.

Remark

- The spectrum of A_k is $\sigma(A_k) = \sigma_p(A_k) = \{if(n)\}_1^\infty = \{\lambda_n\}_1^\infty \subset i\mathbb{R}$, it satisfies

$$\lim_{n\to\infty} i\lambda_n = -\infty, \qquad \lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0,$$

and the corresp. eigenvectors $\{e_n\}_{n=1}^\infty$ are dense and minimal, hence $\overline{D(A_k)} = H_k(\{e_n\})$, but do not form a Schauder basis.

- The resolvent of A_k is given by $(A_k - \lambda I)^{-1} x = (\mathfrak{f}) \sum_{n=1}^{\infty} \frac{c_n e_n}{if(n)-\lambda}$, $\lambda \in \rho(A_k) = \mathbb{C} \setminus \{if(n)\}_1^\infty$, where $x = (\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n \in H_k(\{e_n\})$.

Remark

Note that the sequence $\{f(n)\}_1^\infty$, although satisfies $\lim_{n\to\infty} f(n) = +\infty$, need not to be monotone and the spectrum $\sigma(A_k) = \sigma_p(A_k) = \{if(n)\}_1^\infty$ of operator A_k from our theorem need not to be simple.

Corollary

For each $k \in \mathbb{N}$ the Cauchy problem

$$\begin{cases} \dot{x}(t) = A_k x(t), & t \in \mathbb{R}, \\ x(0) = x_0, \end{cases} \tag{7}$$

with A_k from the above theorem is well-posed, and the solution is given by the formula (5), where $x = x_0$.

Proposition

Let $k \in \mathbb{N}$ and $\left\{e^{A_k t}\right\}_{t\in\mathbb{R}}$ is the C_0-group from the above theorem. Then:

1. $\left\|e^{A_k t}\right\| \to \infty$, when $t \to \pm\infty$.
2. There exists a polynomial $\mathfrak{p}_k$ with positive coefficients, $\deg \mathfrak{p}_k = k$, such that for every $t \in \mathbb{R}$ we have

$$\left\|e^{A_k t}\right\| \leq \mathfrak{p}_k(|t|).$$

So our class of C_0-groups belongs to the class $\mathfrak{P}$ of polynomially bounded C_0-groups studied by T. Eisner, H. Zwart, M. Malejki from 2000's. The class $\mathfrak{P}$, in its turn, belongs to the class of nonquasianalytic groups studied by Yu. I. Lyubic, V. I. Matsaev and V. Q. Phong in 1960's-1990's.
Combining the above proposition with result of T. Eisner & H. Zwart (Semigroup Forum, 2007) we obtain the following.

Proposition

Let $k \in \mathbb{N}$ and A_k is the generator of C_0-group from the above theorem. Then for every $a > 0$ there exists $C > 0$ such that

1. $\left\|(A_k - \lambda I)^{-1}\right\| \leq \frac{C}{|\Re\lambda|^{k+1}}, \quad \text{for all } \lambda : \ 0 < |\Re\lambda| < a;$
2. $\left\|(A_k - \lambda I)^{-1}\right\| \leq C, \quad \text{for all } \lambda : \ |\Re\lambda| \geq a.$

Also we study the questions posed at the beginning

in the Banach space setting and obtain similar answers!

To obtain these results we

- Introduce and study special classes of Banach spaces $\ell_{p,k}(\{e_n\})$, $p \geq 1$, $k \in \mathbb{N}$. Space $\ell_{p,k}(\{e_n\})$ depend on ℓ_p space and a chosen symmetric basis $\{e_n\}_1^\infty$ of ℓ_p.
- Prove that, if $p > 1$, then $\{e_n\}_1^\infty$ is dense and minimal in $\ell_{p,k}(\{e_n\})$ but not uniformly minimal, hence do not form a Schauder basis.
- Consider our classes of increasing sequences $\mathcal{S}_k$, $k \in \mathbb{N}$.

The concept of symmetric basis

was first introduced and studied by I. Singer, Revue de math. pures et appl., 1961, in connection with S. Banach's closed hyperplane problem and related question of C. Bessaga & A. Pelczynski from isomorphic theory of Banach spaces.

A basis $\{\phi_n\}_{n=1}^{\infty}$ of a Banach space X is called symmetric

provided each permutation $\{\phi_{\sigma(n)}\}_{n=1}^{\infty}$ of basis $\{\phi_n\}_{n=1}^{\infty}$ also forms a basis of X, isomorphic to $\{\phi_n\}_{n=1}^{\infty}$.

Example

- The canonical basis of ℓ_p and c_0 space is symmetric.
- The class of symmetric bases in a Hilbert space coincides with the class of Riesz bases.
- The space $L_p(0,1)$, $1 \leq p \neq 2$, does not have a symmetric basis.

It is known that

space ℓ_p, $1 \leq p \leq \infty$, has unique, up to isomorphism, symmetric basis.

So we arrive at the following

Proposition

Let $\{\phi_n\}_{n=1}^{\infty}$ be a basis of ℓ_p, $1 \leq p < \infty$. Then $\{\phi_n\}_{n=1}^{\infty}$ forms a symmetric basis of ℓ_p if and only if there exists constants $M \geq m > 0$ such that for each $x = \sum_{n=1}^{\infty} \alpha_n \phi_n \in \ell_p$ we have

$$m\|x\|^p \leq \sum_{n=1}^{\infty} |\alpha_n|^p \leq M\|x\|^p. \tag{8}$$

This proposition is at the core of generalizations of our results to the case of Banach spaces of special structure.

Spaces $\ell_{p,k}(\{e_n\})$, $p \geq 1$, $k \in \mathbb{N}$

Choose the space ℓ_p and let $\{e_n\}_1^\infty$ be an arbitrary symmetric basis in ℓ_p, $p \geq 1$. Then we define a Banach space $\ell_{p,k}(\{e_n\})$, $p \geq 1$, $k \in \mathbb{N}$, as

$$\ell_{p,k}(\{e_n\}) = \left\{ x = (\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n : \{c_n\}_{n=1}^{\infty} \in \ell_p(\Delta^k) \right\}, \ p \geq 1, \ k \in \mathbb{N},$$

with $\left\| (\mathfrak{f}) \sum\limits_{n=1}^{\infty} c_n e_n \right\|_k = \left\| \sum\limits_{n=1}^{\infty} \left(\Delta^k c_n\right) e_n \right\| = \left\| \sum\limits_{n=1}^{\infty} \sum\limits_{j=0}^{k} (-1)^j C_k^j c_{n-j} e_n \right\|.$

Here $\ell_p(\Delta^k) = \{\alpha = \{\alpha_n\}_{n=1}^{\infty} : \Delta^k \alpha \in \ell_p\}$.

$\ell_{p,k}(\{e_n\})$, $p \geq 1$, $k \in \mathbb{N}$, is isomorphic to ℓ_p and the following holds:

$\ell_p \subset \ell_{p,1}(\{e_n\}) \subset \ell_{p,2}(\{e_n\}) \subset \ell_{p,3}(\{e_n\}) \subset \dots$

Proposition

1. If $p>1$, then $\overline{Lin}\{e_n\}_{n=1}^{\infty}=\ell_{p,k}(\{e_n\})$;
2. $\{e_n\}_{n=1}^{\infty}$ does not form a basis of $\ell_{p,k}(\{e_n\})$;
3. If $p>1$, then $\{e_n\}_{n=1}^{\infty}$ has a unique biorthogonal system
$$\left\{\chi_n=(I-T)^{-k}(I-T^*)^{-k}e_n^*\right\}_{n=1}^{\infty}$$
in $\left(\ell_{p,k}(\{e_n\})\right)^*$, where $Te_n=e_{n+1}$, $n\in\mathbb{N}$, and $\{e_n^*\}_{n=1}^{\infty}$ is biorthogonal to $\{e_n\}_{n=1}^{\infty}$ basis of ℓ_q, where $\frac{1}{p}+\frac{1}{q}=1$;
4. If $p>1$, then $\{\chi_n\}_{n=1}^{\infty}$ is uniformly minimal sequence in $\left(\ell_{p,k}(\{e_n\})\right)^*$ while the sequence $\{e_n\}_{n=1}^{\infty}$ is minimal but not uniformly minimal in $\ell_{p,k}(\{e_n\})$;
5. $L=\left\{x=(\mathfrak{f})\sum_{n=1}^{\infty}c_ne_n\in\ell_{p,k}(\{e_n\}):\ \{c_n\}_{n=1}^{\infty}\in\ell_p(\Delta^k)\cap c_0\right\}$ is not a (closed) subspace of $\ell_{p,k}(\{e_n\})$.

Theorem

Let $\{e_n\}_{n=1}^{\infty}$ be a symmetric basis of ℓ_p, $p>1$ and $k\in\mathbb{N}$. Then $\{e_n\}_{n=1}^{\infty}$ does not form a Schauder basis of $\ell_{p,k}(\{e_n\})$ and the operator $A_k:\ell_{p,k}(\{e_n\})\supset D(A_k)\mapsto \ell_{p,k}(\{e_n\})$, defined by

$$A_k x = A_k(\mathfrak{f})\sum_{n=1}^{\infty} c_n e_n = (\mathfrak{f})\sum_{n=1}^{\infty} if(n)\cdot c_n e_n,$$

where $\{f(n)\}_{n=1}^{\infty}\in\mathcal{S}_k$, with domain

$$D(A_k)=\left\{x=(\mathfrak{f})\sum_{n=1}^{\infty} c_n e_n\in \ell_{p,k}(\{e_n\}):\ \{f(n)\cdot c_n\}_{n=1}^{\infty}\in \ell_p(\Delta^k)\right\},$$

generates the C_0-group $\{e^{A_k t}\}_{t\in\mathbb{R}}$ on $\ell_{p,k}(\{e_n\})$, which is given by

$$e^{A_k t}x = e^{A_k t}(\mathfrak{f})\sum_{n=1}^{\infty} c_n e_n = (\mathfrak{f})\sum_{n=1}^{\infty} e^{itf(n)} c_n e_n. \tag{9}$$

Multiple application of the discrete Hardy inequality for $p > 1$

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^p \leq \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n^p$$

plays the key role in the proof of this theorem.

Remark

- The spectrum of A_k is $\sigma_p(A_k) = \{if(n)\}_1^\infty = \{\lambda_n\}_1^\infty \subset i\mathbb{R}$, it satisfies

$$\lim_{n\to\infty} i\lambda_n = -\infty, \qquad \lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0,$$

  and the corresp. eigenvectors $\{e_n\}_{n=1}^\infty$ are dense and minimal, but do not form a Schauder basis.
- The resolvent of A_k is given by $(A_k - \lambda I)^{-1} x = (\mathfrak{f}) \sum_{n=1}^{\infty} \frac{c_n e_n}{if(n)-\lambda}$,

  $\lambda \in \rho(A_k) = \mathbb{C} \setminus \{if(n)\}_1^\infty$, where $x = (\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n \in \ell_{p,k}(\{e_n\})$.

The proof of this theorem is similar to the proof of infinitesimality result in spaces $H_k(\{e_n\})$, $k \in \mathbb{N}$, and the ingredients are the same.

Corollary

For each $k \in \mathbb{N}$ the Cauchy problem

$$\begin{cases} \dot{x}(t) = A_k x(t), & t \in \mathbb{R}, \\ x(0) = x_0, \end{cases} \tag{10}$$

with A_k from the above theorem is well-posed, and the solution is given by the formula (9), where $x = x_0$.

Proposition

The class of C_0-groups $\{e^{A_k t}\}_{t \in \mathbb{R}}$ from the above theorem also belongs to the class of polynomially bounded C_0-groups.

Proposition

Let $\{\lambda_n\}_{n=1}^{\infty} \subset i\mathbb{R}$ satisfy

$$\lim_{n\to\infty} i\lambda_n = -\infty, \qquad \lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0,$$

and $\exists \alpha \in \left(0, \frac{1}{2}\right] : \liminf_{n\to\infty} n^{\alpha}|\lambda_n - \lambda_{n-1}| > 0$. Then the operator A, defined by $Ax = A(\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n = (\mathfrak{f}) \sum_{n=1}^{\infty} \lambda_n c_n e_n$, with domain $D(A) = \left\{ x = (\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n \in H_1(\{e_n\}) : \{\lambda_n c_n\}_{n=1}^{\infty} \in \ell_2(\Delta) \right\}$, does not generate the C_0-semigroup on the space $H_1(\{e_n\})$.

Example

We can take $\lambda_n = i\sqrt{n}$, $n \in \mathbb{N}$.

Corollary

The Cauchy problem (1) with A from the proposition above is ill-posed on $H_1(\{e_n\})$.

Open questions:

- Is it possible to construct the unbounded generator of the C_0-group with purely imaginary eigenvalues not satisfying (3) and family of eigenvectors, which form a bounded non-Riesz basis in a Hilbert space?
- What natural evolution phenomena are described by a such kind of evolution equations?
- What happens between $i \ln n$ and $i\sqrt{n}$ in our constructions in $H_l(\{e_n\})$?
- How can the spectral theorem of G.Q. Xu & S.P. Yung, and H. Zwart be generalized to the case of some kind of bases in Banach spaces, e.g. symmetric bases?

Thanks for the attention!