Spectral Theory, Differential Equations and Probability
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Homogenization in domains with traps. Part 2

Spectral properties of domains with ”room-and-passage” boundary

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CRC 1173 Wave phenomena

Joint work with Giuseppe Cardone (University of Sannio, Benevento, Italy)

Outline of the talk:

- Preliminaries
- Results: bounded domain
- Results: fix width strip
- Results: thin strip

Let Ω be a bounded domain in $\mathbb{R}^n$. We perturb it to a family of domains $\{\Omega^\varepsilon \subset \mathbb{R}^n\}_\varepsilon$, here $\varepsilon > 0$ is a small parameter.

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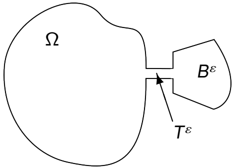
In general, however, this is not true – even if Ω^ε differs from Ω only in a ball of the radius $\mathcal{O}(\varepsilon)$.

The example below demonstrates this.

Example by R. Courant and D. Hilbert

Let $\varepsilon > 0$ be a small parameter. We set:

- $\Omega \subset \mathbb{R}^2$ – fixed bounded domain
- $B^\varepsilon \cong \varepsilon B$, B is a fixed bounded domain – room
- $T^\varepsilon \cong [0, h^\varepsilon] \times (0, d^\varepsilon)$ – passage
- $\Omega^\varepsilon = \Omega \cup (T^\varepsilon \cup B^\varepsilon)$ – perturbed domain



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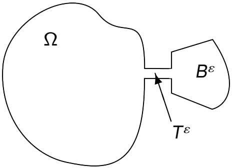

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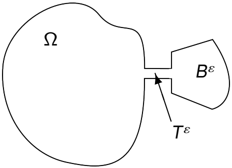

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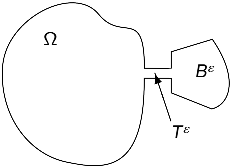

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$$\lim_{\varepsilon\to 0} \lambda_2(\Omega^\varepsilon) = 0 \text{ provided } h^\varepsilon = \varepsilon,\ d^\varepsilon = \varepsilon^\alpha, \alpha > 3$$

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Our goal is to extend these results under weaker restrictions on sizes of "rooms" and "passages" and with an additional "mass" inside the "rooms".

Remark 1: General result for the Neumann Laplacian

M. Lobo-Hidalgo, E. Sanchez-Palencia, Comm. PDEs 4 (1979)

Let $\Omega \subset \mathbb{R}^n$ be a fixed domain and let $\{\Omega^\varepsilon \subset \mathbb{R}^n\}_\varepsilon$ be a family of domains satisfying some mild regularity assumptions and

$$\Omega \subset \Omega^\varepsilon, \quad |\Omega^\varepsilon \setminus \Omega| \to 0 \text{ as } \varepsilon \to 0, \tag{$\star$}$$

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$$\Omega \subset \Omega^\varepsilon, \quad |\Omega^\varepsilon \setminus \Omega| \to 0 \text{ as } \varepsilon \to 0, \qquad (\star)$$

where $|\cdot|$ stays for the Lebesque measure in $\mathbb{R}^n$. Then

$$\forall \lambda \in \sigma(-\Delta_\Omega)\ \exists \lambda^\varepsilon \in \sigma(-\Delta_{\Omega^\varepsilon}) :\ \lim_{\varepsilon\to 0} \lambda^\varepsilon = \lambda.$$

However, it may happen that $(\star)$ holds, but

$$\exists \lambda^\varepsilon \in \sigma(-\Delta_{\Omega^\varepsilon}),\ \lambda^\varepsilon \to \lambda \text{ as } \varepsilon \to 0 \text{ and } \lambda \notin \sigma(-\Delta_\Omega). \qquad (\star\star)$$

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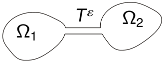

In a simplest case they are defined as follows: let Ω be a union of two disjoint domains Ω_j, $j = 1, 2$ and $\Omega^\varepsilon = \Omega \cup T^\varepsilon$, where T^ε is a narrow channel connecting Ω_1 and Ω_2 and approaching as $\varepsilon \to 0$ an 1-dimensional line segment of the length h.

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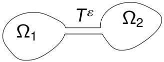

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One can prove that if $\sigma(-\Delta_{\Omega^\varepsilon}) \ni \lambda^\varepsilon \to \lambda$ as $\varepsilon \to 0$ then either $\lambda \in \sigma(-\Delta_{\Omega_1}) \cup \sigma(-\Delta_{\Omega_2})$ or $\lambda = \left(\frac{\pi k}{h}\right)^2$ for some $k \in \mathbb{N}$.

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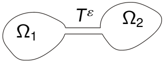

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- C. Anné, Proc. Amer. Math. Soc. 123 (1995)
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Domains with "room-and-passage"-like geometry are widely used in order to construct examples illustrating various phenomena in Sobolev spaces theory and in spectral theory.

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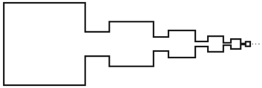

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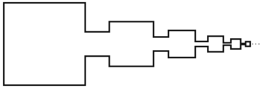

More details: [V. Maz'ya, Sobolev spaces with applications to elliptic partial differential equations, Springer, 2011].

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i. for every compact set $F \subset \Omega$ there is $\varepsilon_0 > 0$ such that $F \subset \Omega^\varepsilon$ provided $\varepsilon < \varepsilon_0$,
ii. for every open set $O \supset \overline{\Omega}$ there is $\varepsilon_0 > 0$ such that $\Omega^\varepsilon \subset O$ provided $\varepsilon < \varepsilon_0$

(plus some mild regularity assumptions on Ω^ε and Ω), then the k-th eigenvalue of the Dirichlet Laplacian in Ω^ε converges to the k-th eigenvalue of the Dirichlet Laplacian in Ω.

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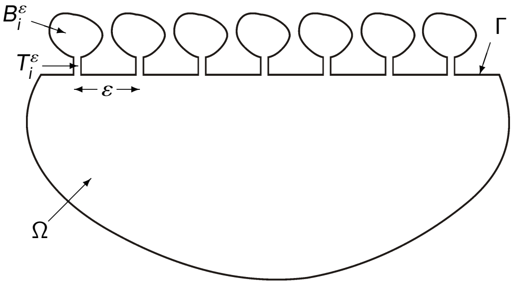
- I. Babuška, R. Vyborny, Czech. Math. J. 15 (1965)
- J. Rauch, M. Taylor, J. Funct. Anal. 18 (1975)



- $\Omega \subset \mathbb{R}^n$ – bounded domain, Γ – flat part of $\partial\Omega$
- $B_i^\varepsilon \cong \varepsilon B$, where $B \subset \mathbb{R}^n$ – rooms
- $T_i^\varepsilon \cong d^\varepsilon D \times [0, h^\varepsilon]$, where $D \subset \mathbb{R}^{n-1}$, d^ε, $h^\varepsilon > 0$ – passages

$$\Omega^\varepsilon = \Omega \cup \left(\bigcup_i (T_i^\varepsilon \cup B_i^\varepsilon) \right)$$

The main object of our interest is the following operator:

$$\mathcal{A}^\varepsilon = -\frac{1}{\rho^\varepsilon}\Delta_{\Omega^\varepsilon}$$

acting in $L_2(\Omega^\varepsilon, \rho^\varepsilon dx)$. Here $-\Delta_{\Omega^\varepsilon}$ is the Neumann Laplacian in Ω^ε, the function ρ^ε (mass density) is defined as follows:

$$\rho^\varepsilon(x) = \begin{cases} \varrho^\varepsilon, & x \in \bigcup\limits_i B_i^\varepsilon \quad \textit{(the union of the rooms)}, \\ 1, & x \in \Omega \cup \left(\bigcup\limits_i T_i^\varepsilon\right). \end{cases}$$

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Our goal

To describe the behaviour of $\sigma(\mathcal{A}^\varepsilon)$ as $\varepsilon \to 0$.

(i) $h^\varepsilon \to 0$ as $\varepsilon \to 0$

(ii) $\lim\limits_{\varepsilon\to 0} \varepsilon \ln d^\varepsilon = 0$ (for $n = 2$) or $\lim\limits_{\varepsilon\to 0} \varepsilon^{n-1}/(d^\varepsilon)^{n-2} = 0$ (for $n > 2$)

(iii) the following limits exist:

$$\lim_{\varepsilon\to 0} \frac{(d^\varepsilon)^{n-1}|D|}{\varrho^\varepsilon h^\varepsilon \varepsilon^n |B|} =: q \in [0,\infty], \quad \lim_{\varepsilon\to 0} \varrho^\varepsilon \varepsilon |B| =: r \in [0,\infty).$$

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The finiteness of r implies the uniform (with respect to ε) boundedness of the total mass m_B^ε of the "rooms":

$$m_B^\varepsilon := \int\limits_{\bigcup_i B_i^\varepsilon} \rho^\varepsilon dx = \varrho^\varepsilon \sum_i |B_i^\varepsilon| = \varrho^\varepsilon \varepsilon |B| \sum_i \varepsilon^{n-1} \sim r|\Gamma|$$

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Remark 2

In the case $\varrho^\varepsilon = 1$ (i.e. $\mathcal{A}^\varepsilon = -\Delta_{\Omega^\varepsilon}$) one has $r = 0$.

Let $r > 0$. By $\mathcal{H}$ we denote the Hilbert space of functions from $L_2(\Omega) \times L_2(\Gamma)$ endowed with the scalar product

$$(U, V)_{\mathcal{H}} = \int_{\Omega} u_1(x)\overline{v_1(x)}\mathrm{d}x + r\int_{\Gamma} u_2(x)\overline{v_2(x)}\mathrm{d}s$$

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By $\mathfrak{a}^0$ we denote the following sesquilinear form in $\mathcal{H}$:

$$\mathfrak{a}^0[U, V] := \int_{\Omega} \nabla u_1 \cdot \nabla\overline{v_1}\mathrm{d}x + qr\int_{\Gamma}(u_1 - u_2)(\overline{v_1 - v_2})\mathrm{d}s$$

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with $\mathrm{dom}(\mathfrak{a}^0) = H^1(\Omega) \times L_2(\Gamma)$.

By $\mathcal{A}^0$ we denote the self-adjoint operator acting in $\mathcal{H}$ being associated with this form.

Formally, the eigenvalue problem

$$\mathcal{A}^0 U = \lambda U,$$

where $U = (u_1, u_2)$, can be written as follows:

$$\begin{cases} -\Delta u_1 = \lambda u_1 & \text{in } \Omega, \\ \dfrac{\partial u_1}{\partial n} + qr(u_1 - u_2) = 0 & \text{on } \Gamma, \\ q(u_2 - u_1) = \lambda u_2 & \text{on } \Gamma, \\ \dfrac{\partial u_1}{\partial n} = 0 & \text{on } \partial\Omega \setminus \Gamma, \end{cases}$$

where n is the outward-pointing unit normal to $\partial\Omega$.

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Equivalently,

$$\begin{cases} -\Delta u_1 = \lambda u_1 & \text{in } \Omega, \\ \dfrac{\partial u_1}{\partial n} = \dfrac{qr\lambda}{q-\lambda} u_1 & \text{on } \Gamma, \\ \dfrac{\partial u_1}{\partial n} = 0 & \text{on } \partial\Omega \setminus \Gamma, \end{cases} \qquad u_2 = \frac{q}{q-\lambda} u_1.$$

Lemma

One has

$$\sigma_{\rm disc}(\mathcal{A}^0) = \{\lambda_k^-, k = 1,2,3\dots\} \cup \{\lambda_k^+, k = 1,2,3\dots\},$$
$$\sigma_{\rm ess}(\mathcal{A}^0) = \{q\},$$

where

$$0 = \lambda_1^- \leq \lambda_2^- \leq \dots \leq \lambda_k^- \leq \dots \underset{k\to\infty}{\to} q < \lambda_1^+ \leq \lambda_2^+ \leq \dots \leq \lambda_k^+ \leq \dots \underset{k\to\infty}{\to} \infty.$$

Theorem 1

Let $q < \infty$, $r > 0$. Let $I \subset \mathbb{R}$ be an arbitrary compact interval.

Then the set $\sigma(\mathcal{A}^\varepsilon) \cap I$ converges in the Hausdorff sense as $\varepsilon \to 0$ to the set $\sigma(\mathcal{A}^0) \cap I$, i.e.

$$\mathrm{dist}_H\left(\sigma(\mathcal{A}^\varepsilon) \cap I, \sigma(\mathcal{A}^0) \cap I\right) \to 0 \text{ as } \varepsilon \to 0.$$

where $\mathrm{dist}_H(X, Y) := \max\left\{\sup_{x\in X} \inf_{y\in Y} |x - y|; \sup_{y\in Y} \inf_{x\in X} |y - x|\right\}$.

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Remark

The claim of the theorem is equivalent to the fulfilment of the conditions

(i) if $\lambda^\varepsilon \in \sigma(\mathcal{A}^\varepsilon)$ and $\lim\limits_{\varepsilon=\varepsilon_k\to 0} \lambda^\varepsilon = \lambda$ then $\lambda \in \sigma(\mathcal{A}^0)$,

(ii) for any $\lambda \in \sigma(\mathcal{A}^0)$ there is $\lambda^\varepsilon \in \sigma(\mathcal{A}^\varepsilon)$ such that $\lim\limits_{\varepsilon\to 0} \lambda^\varepsilon = \lambda$.

Theorem 2

Let $q < \infty$, $r = 0$. Let $I \subset \mathbb{R}$ be an arbitrary compact interval.

Then the set $\sigma(\mathcal{A}^\varepsilon) \cap I$ converges in the Hausdorff sense as $\varepsilon \to 0$ to the set

$$\Big(\sigma(-\Delta_\Omega) \cup \{q\}\Big) \cap I.$$

By $\widetilde{\mathfrak{a}}^0$ we denote the following sesquilinear form in the space $\mathcal{H}$ (recall: $\mathcal{H} = L_2(\Omega) \times L_2(\Gamma, r\mathrm{d}s)$):

$$\widetilde{\mathfrak{a}}^0[U, V] := \int\limits_\Omega \nabla u_1 \cdot \nabla \overline{v_1} \mathrm{d}x$$

with $\widetilde{\mathfrak{a}}^0 = \left\{U \in H^1(\Omega) \times L_2(\Gamma) : \ u_1|_\Gamma = u_2\right\}$. By $\widetilde{\mathcal{A}}^0$ we denote the self-adjoint operator acting in $\mathcal{H}$ being associated with this form.

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Formally, the eigenvalue problem $\widetilde{\mathcal{A}}^0 U = \lambda U$, where $U = (u_1, u_2)$, can be written as follows:

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = \lambda r u & \text{on } \Gamma, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \setminus \Gamma. \end{cases}$$

Theorem 3

Let $q = \infty$, $r > 0$. Let $I \subset \mathbb{R}$ be an arbitrary compact interval.

Then the set $\sigma(\mathcal{A}^\varepsilon) \cap I$ converges in the Hausdorff sense as $\varepsilon \to 0$ to the set

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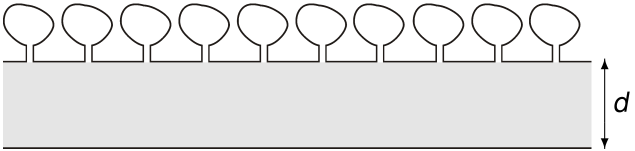
Theorem 4

Let $q = \infty,\ r = 0$. Let $I \subset \mathbb{R}$ be an arbitrary compact interval.

Then the set $\sigma(\mathcal{A}^\varepsilon) \cap I$ converges in the Hausdorff sense as $\varepsilon \to 0$ to the set

$$\sigma(-\Delta_\Omega) \cap I.$$

$$\Omega = \mathbb{R} \times (0, d), \quad \Gamma = \{x \in \mathbb{R}^2 : \ x_2 = d\}, \quad d > 0$$



$$\Omega^\varepsilon = \Omega \cup \left(\bigcup_{i \in \mathbb{Z}} (T_i^\varepsilon \cup B_i^\varepsilon) \right)$$

$$\lim_{\varepsilon\to 0}\frac{(d^\varepsilon)^{n-1}|D|}{h^\varepsilon\varepsilon^n|B|}=:q,\quad \lim_{\varepsilon\to 0}\varrho^\varepsilon\varepsilon|B|=:r.$$

We focus on the case $r, q > 0$.

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Let $I \subset \mathbb{R}$ be an arbitrary compact interval. Then the set $\sigma(\mathcal{A}^\varepsilon)\cap I$ converges in the Hausdorff sense as $\varepsilon\to 0$ to the set $\sigma(\mathcal{A}^0)\cap I$.

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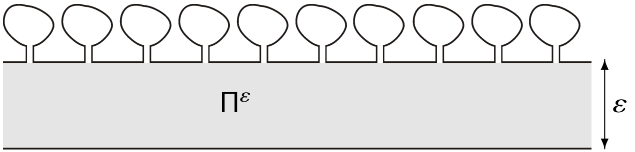
Let $I \subset \mathbb{R}$ be an arbitrary compact interval. Then the set $\sigma(\mathcal{A}^\varepsilon) \cap I$ converges in the Hausdorff sense as $\varepsilon \to 0$ to the set $\sigma(\mathcal{A}^0) \cap I$.

The spectrum of the operator $\mathcal{A}^0$ has the following form:

$$\sigma(\mathcal{A}^0) = \begin{cases} [0, q] \cup [\widehat{q}, \infty), & q < \left(\frac{\pi}{2d}\right)^2 \\ [0, \infty), & q \geq \left(\frac{\pi}{2d}\right)^2 \end{cases}.$$

Here $\widehat{q}$ is some number satisfying $q < \widehat{q} < \left(\frac{\pi}{2d}\right)^2$.

- $\Pi^\varepsilon \cong \mathbb{R} \times (0, \varepsilon) \subset \mathbb{R}^2$ – straight strip of the width ε
- $B_i^\varepsilon \cong \varepsilon B$, $B \subset \mathbb{R}^2$ – rooms ($i \in \mathbb{Z}$)
- $T_i^\varepsilon \cong (0, d^\varepsilon) \times [0, h^\varepsilon]$, $d^\varepsilon, h^\varepsilon > 0$ – passages ($i \in \mathbb{Z}$)



$$\Omega^\varepsilon = \Pi^\varepsilon \cup \left(\bigcup_{i \in \mathbb{Z}} (T_i^\varepsilon \cup B_i^\varepsilon) \right)$$

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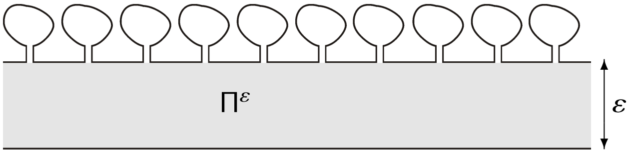

$$\Omega^\varepsilon = \Pi^\varepsilon \cup \left(\bigcup_{i \in \mathbb{Z}} (T_i^\varepsilon \cup B_i^\varepsilon) \right)$$

We denote by $\mathcal{A}^\varepsilon = -\Delta_{\Omega^\varepsilon}$ the Neumann Laplacian in Ω^ε.

We suppose that the following conditions hold as $\varepsilon \to 0$:

(i) $d^\varepsilon = o(\varepsilon)$

(ii) $\varepsilon^2 \ln d^\varepsilon \to 0$

(iii) $h^\varepsilon \to 0$

(iv) the following limit exists and is positive:

$$\lim_{\varepsilon \to 0} \frac{d^\varepsilon}{h^\varepsilon \varepsilon^2 |B|} =: q \in (0, \infty).$$

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(iv) the following limit exists and is positive:

$$\lim_{\varepsilon\to 0} \frac{d^\varepsilon}{h^\varepsilon \varepsilon^2 |B|} =: q \in (0, \infty).$$

Also, we denote $r := |B|$.

By $\mathcal{H}$ we denote the Hilbert space of functions from $L_2(\mathbb{R}) \times L_2(\mathbb{R})$ endowed with the scalar product

$$(U, V)_{\mathcal{H}} = \int\limits_{\mathbb{R}} u_1(x)\overline{v_1(x)}\mathrm{d}x + r\int\limits_{\mathbb{R}} u_2(x)\overline{v_2(x)}\mathrm{d}x$$

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By $\mathfrak{a}^0$ we denote the following sesquilinear form in $\mathcal{H}$:

$$\mathfrak{a}^0[U, V] := \int\limits_{\mathbb{R}} \nabla u_1 \cdot \nabla\overline{v_1}\mathrm{d}x + qr\int\limits_{\mathbb{R}} (u_1 - u_2)(\overline{v_1 - v_2})\mathrm{d}x$$

with $\mathrm{dom}(\mathfrak{a}^0) = H^1(\mathbb{R}) \times L_2(\mathbb{R})$.

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By $\mathcal{A}^0$ we denote the self-adjoint operator acting in $\mathcal{H}$ being associated with this form.

$$\mathcal{A}^0 U = \lambda U$$

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$$\begin{cases} -u_1'' + qr(u_1 - u_2) = \lambda u_1, \\ q(u_2 - u_1) = \lambda u_2. \end{cases}$$

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$$\begin{cases} -u_1'' = \lambda \left(1 + \frac{qr}{q-\lambda}\right) u_1, \\ u_2 = u_1 \frac{q}{q-\lambda}. \end{cases}$$

Theorem

Let $I \subset \mathbb{R}$ be an arbitrary compact interval. Then the set $\sigma(\mathcal{A}^\varepsilon) \cap I$ converges in the Hausdorff sense as $\varepsilon \to 0$ to the set $\sigma(\mathcal{A}^0) \cap I$.

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The spectrum of the operator $\mathcal{A}^0$ has the following form:

$$\sigma(\mathcal{A}^0) = [0, \infty) \setminus (q, \widehat{q}),$$

where $\widehat{q} = q + qr$.

Remark 4: Another examples of waveguides with gaps

- K. Yoshitomi (1998)
- P. Exner, O. Post (2005)
- L. Friedlander, M. Solomyak (2008)
- S. Nazarov (2009-...),
- G. Cardone, V. Minutolo, S. Nazarov (2009)
- S. Nazarov, G. Cardone, C. Perugia (2010)
- S. Nazarov, K. Taskinen (2013)
- F. Bakharev, S. Nazarov, S. Ruotsalainen (2013)
- D. Borisov, K. Pankrashkin (2013)




Let $m \in \mathbb{N}$ be arbitrary.

In order to open up m gaps we attach m families of RP domains:

$$\Omega^\varepsilon = \Pi^\varepsilon \cup \left(\bigcup_{i\in\mathbb{Z}} \bigcup_{j=1}^{m} T_{ij}^\varepsilon \cup B_{ij}^\varepsilon \right),$$

where $T_{ij} \cong (0, d_j^\varepsilon) \times [0, h_j^\varepsilon]$, $B_{ij}^\varepsilon \cong \varepsilon B_j$. Here $d_j^\varepsilon > 0$, $h_j^\varepsilon > 0$, $B_j \subset \mathbb{R}^2$.

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Under the same assumptions as in the case $m = 1$ one has:

- the operator $\mathcal{A}^\varepsilon$ has at least m gaps as ε is small enough,

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- the operator $\mathcal{A}^\varepsilon$ has at least m gaps as ε is small enough,
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Under the same assumptions as in the case $m = 1$ one has:

- the operator $\mathcal{A}^\varepsilon$ has at least m gaps as ε is small enough,
- the first m gaps converge as $\varepsilon \to 0$ to certain intervals (a_j, b_j), whose closures are pairwise disjoint; the next gaps (if any) go to infinity,
- one can completely control the location of the intervals (a_j, b_j) via a suitable choice of the numbers d_j^ε, h_j^ε and the domains B_j.

References

- G.Cardone, A.K., J. Differ. Equations 259(6) (2015)
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- G.Cardone, A.K., submitted; arXiv:1608.00440 (2016)

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- G.Cardone, A.K., J. Differ. Equations 259(6) (2015)
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Thank you for the attention!