Homogenization of Hyperbolic-type Equations with Periodic Coefficients

Mark Dorodnyi and Tatiana Suslina

St. Petersburg State University

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Statement of the problem

Let Γ be a lattice in $\mathbb{R}^d$, let Ω be the cell of Γ. Let $\widetilde{\Gamma}$ be the dual lattice. By $\widetilde{\Omega}$ we denote the central Brillouin zone of $\widetilde{\Gamma}$.

Example:

$$\Gamma = \mathbb{Z}^d, \quad \Omega = (0,1)^d\,; \quad \widetilde{\Gamma} = (2\pi\mathbb{Z})^d, \quad \widetilde{\Omega} = (-\pi,\pi)^d.$$

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Main object

In $L_2(\mathbb{R}^d;\mathbb{C}^n)$, we consider elliptic second order DO

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Here $g(\mathbf{x})$ is Hermitian $(m \times m)$-matrix with complex entries. We assume that $g(\mathbf{x})$ is Γ-periodic, bounded and positive definite:

$$c' \mathbf{1}_m \leqslant g(\mathbf{x}) \leqslant c'' \mathbf{1}_m, \quad 0 < c'' \leqslant c'' < \infty.$$

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$b(\mathbf{D}) = \sum_{j=1}^d b_j D_j$ is a first order $(m \times n)$-matrix DO; b_j are constant matrices, and $m \geqslant n$. The symbol $b(\boldsymbol{\xi}) = \sum_{j=1}^d b_j \xi_j$ is such that

$$\operatorname{rank} b(\boldsymbol{\xi}) = n, \quad 0 \neq \boldsymbol{\xi} \in \mathbb{R}^d.$$

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Precise Definition: A_ε is a selfadjoint operator in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ generated by the quadratic form

$$a_\varepsilon[\mathbf{u},\mathbf{u}] = \int_{\mathbb{R}^d} \langle g^\varepsilon(\mathbf{x}) b(\mathbf{D})\mathbf{u}, b(\mathbf{D})\mathbf{u}\rangle \, d\mathbf{x}, \quad \mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n).$$

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Under our assumptions,

$$c_0 \int_{\mathbb{R}^d} |\mathbf{D}\mathbf{u}|^2 d\mathbf{x} \leqslant a_\varepsilon[\mathbf{u},\mathbf{u}] \leqslant c_1 \int_{\mathbb{R}^d} |\mathbf{D}\mathbf{u}|^2 \, d\mathbf{x}, \quad \mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n).$$

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Example: $A_\varepsilon = -\operatorname{div} g^\varepsilon(\mathbf{x})\nabla = \mathbf{D}^* g^\varepsilon(\mathbf{x})\mathbf{D}$.

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Problem

The problem is to study the behavior of the operator

$$\cos(\tau A_\varepsilon^{1/2}), \ \tau \in \mathbb{R},$$

for small ε, and to apply the results to the Cauchy problem for the hyperbolic-type equation:

$$\begin{cases} \partial_\tau^2 \mathbf{u}_\varepsilon(\mathbf{x},\tau) = -(A_\varepsilon \mathbf{u}_\varepsilon)(\mathbf{x},\tau), \quad \mathbf{x} \in \mathbb{R}^d, \quad \tau \in \mathbb{R}, \\ \mathbf{u}_\varepsilon(\mathbf{x},0) = \boldsymbol{\phi}(\mathbf{x}), \quad \partial_\tau \mathbf{u}_\varepsilon(\mathbf{x},0) = \boldsymbol{\psi}(\mathbf{x}). \end{cases}$$

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We show that, in some sense,

$$\cos(\tau A_\varepsilon^{1/2}) \sim \cos(\tau (A^0)^{1/2}), \quad \varepsilon \to 0,$$

where A^0 is the effective operator with constant effective coefficients.

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Let $\Lambda(\mathbf{x})$ be the $(n \times m)$-matrix-valued Γ-periodic solution of the problem

$$b(\mathbf{D})^* g(\mathbf{x})(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m) = 0; \quad \int_\Omega \Lambda(\mathbf{x})\, d\mathbf{x} = 0.$$

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Then g^0 is an $(m \times m)$-matrix given by

$$g^0 = |\Omega|^{-1} \int_\Omega \widetilde{g}(\mathbf{x})\, d\mathbf{x}, \quad \widetilde{g}(\mathbf{x}) := g(\mathbf{x})(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m).$$

Survey

- In 2001 **M. Birman and T. Suslina** proved that

$$\|(A_\varepsilon + I)^{-1} - (A^0 + I)^{-1}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant C\varepsilon. \qquad (1)$$

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- A different approach to operator error estimates was suggested by **V. Zhikov and S. Pastukhova** (2005–2006).

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$$\| \exp(-i\tau A_\varepsilon) - \exp(-i\tau A^0) \|_{H^3(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leqslant (\widetilde{\mathcal{C}}_1 + \widetilde{\mathcal{C}}_2 |\tau|)\varepsilon,$$
$$\| \cos(\tau A_\varepsilon^{1/2}) - \cos(\tau (A^0)^{1/2}) \|_{H^2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leqslant (\mathcal{C}_1 + \mathcal{C}_2 |\tau|)\varepsilon. \quad (3)$$

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Answers

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Answers

YES!, YES!

Similar results were obtained also for nonstationary Schrödinger type equations by **T.A. Suslina**.

Reduction 1: Scaling transformation

Question: for what (minimal) s the estimate

$$\| e^{-i\tau A_\varepsilon^{1/2}} - e^{-i\tau (A^0)^{1/2}} \|_{H^s(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leqslant C(\tau)\varepsilon \qquad (4)$$

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holds? Let $H_0 = -\Delta$. Clearly, (4) is equivalent to

$$\left\| \left(e^{-i\tau A_\varepsilon^{1/2}} - e^{-i\tau (A^0)^{1/2}} \right) (H_0 + I)^{-s/2} \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leqslant C(\tau)\varepsilon. \qquad (5)$$

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Next, by the scaling transformation, (5) is equivalent to

$$\left\|\left(e^{-i\tau\varepsilon^{-1} A^{1/2}} - e^{-i\tau\varepsilon^{-1} (A^0)^{1/2}}\right)\varepsilon^s(H_0 + \varepsilon^2 I)^{-s/2}\right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant C(\tau)\varepsilon. \qquad (6)$$

Here $A = b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D})$.

Reduction 2: Direct integral expansion

Using the Floquet-Bloch theory, we expand the operators A, A^0 and H_0 (acting in $L_2(\mathbb{R}^d;\mathbb{C}^n)$) in the direct integrals:

$$A \sim \int_{\widetilde{\Omega}} \oplus A(\mathbf{k})\, d\mathbf{k}, \quad A^0 \sim \int_{\widetilde{\Omega}} \oplus A^0(\mathbf{k})\, d\mathbf{k}, \quad H_0 \sim \int_{\widetilde{\Omega}} \oplus H_0(\mathbf{k})\, d\mathbf{k}.$$

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The parameter $\mathbf{k} \in \widetilde{\Omega}$ is called the *quasimomentum*. The operators $A(\mathbf{k})$, $A^0(\mathbf{k})$ and $H_0(\mathbf{k})$ act in $L_2(\Omega;\mathbb{C}^n)$ and are defined by the expressions

$$\begin{aligned} A(\mathbf{k}) &= b(\mathbf{D}+\mathbf{k})^* g(\mathbf{x}) b(\mathbf{D}+\mathbf{k}), \\ A^0(\mathbf{k}) &= b(\mathbf{D}+\mathbf{k})^* g^0 b(\mathbf{D}+\mathbf{k}), \\ H_0(\mathbf{k}) &= |\mathbf{D}+\mathbf{k}|^2 \end{aligned}$$

with periodic boundary conditions. The precise definitions are given in terms of the corresponding quadratic forms.

Reduction 2: Direct integral expansion

Using the direct integral expansions, we see that estimate

$$\left\| \left(e^{-i\tau\varepsilon^{-1}A^{1/2}} - e^{-i\tau\varepsilon^{-1}(A^0)^{1/2}} \right) \varepsilon^s (H_0 + \varepsilon^2 I)^{-s/2} \right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant C(\tau)\varepsilon \tag{6}$$

is equivalent to estimate

$$\left\| \left(e^{-i\tau\varepsilon^{-1}A(\mathbf{k})^{1/2}} - e^{-i\tau\varepsilon^{-1}A^0(\mathbf{k})^{1/2}} \right) \varepsilon^s (H_0(\mathbf{k}) + \varepsilon^2 I)^{-s/2} \right\|_{L_2(\Omega)\to L_2(\Omega)} \leqslant \leqslant C(\tau)\varepsilon, \tag{7}$$

for almost every $\mathbf{k} \in \widetilde{\Omega}$.

Reduction 3: Projection onto the subspace of constant vector-valued functions

Next, let P be the projection onto the subspace

$$\mathfrak{N} = \{\mathbf{u} \in L_2(\Omega; \mathbb{C}^n) : \mathbf{u} = \mathbf{c} \in \mathbb{C}^n\}.$$

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It is easily seen that for $s \geqslant 1$ estimate (7) is equivalent to

$$\left\| \left(e^{-i\tau\varepsilon^{-1} A(\mathbf{k})^{1/2}} - e^{-i\tau\varepsilon^{-1} S(\mathbf{k})^{1/2}} \right) P \right\|_{L_2(\Omega)\to L_2(\Omega)} \varepsilon^s (|\mathbf{k}|^2 + \varepsilon^2)^{-s/2} \leqslant C(\tau)\varepsilon \tag{8}$$

for $|\mathbf{k}| \leqslant t^0$. Here

$$S(\mathbf{k}) = b(\mathbf{k})^* g^0 b(\mathbf{k}).$$

Analytic perturbation theory

The operator $A(\mathbf{k})$ acting in $L_2(\Omega; \mathbb{C}^n)$ is an elliptic operator in a bounded domain; its spectrum is discrete. This operator depends on $\mathbf{k}$ analytically. We consider $A(0)$ as an unperturbed operator and $A(\mathbf{k})$ as a perturbed operator.

Analytic perturbation theory

For $\mathbf{k} = 0$ the operator $A(0)$ (given by $b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D})$ with periodic boundary conditions) has a kernel $\mathfrak{N}$ consisting of constant vector-valued functions:

$$\operatorname{Ker} A(0) = \mathfrak{N} = \{\mathbf{u} \in L_2(\Omega; \mathbb{C}^n) \colon \mathbf{u} = \mathbf{c} \in \mathbb{C}^n\}.$$

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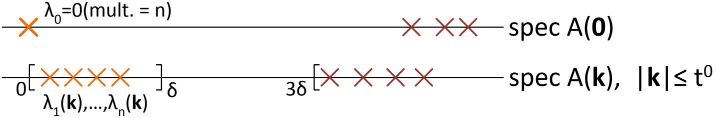

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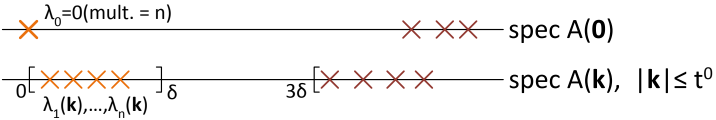

Only these eigenvalues $\lambda_1(\mathbf{k}), \dots, \lambda_n(\mathbf{k})$ (and the corresponding eigenfunctions) are important for our problem.

Analytic perturbation theory

We put

$$\mathbf{k} = t\boldsymbol{\theta}, \quad t = |\mathbf{k}|, \quad \boldsymbol{\theta} \in \mathbb{S}^{d-1}.$$

and study the operator family $A(\mathbf{k}) = A(t\boldsymbol{\theta}) =: A(t, \boldsymbol{\theta})$ by methods of the analytic perturbation theory with respect to the one-dimentional parameter t. But we have to make our constructions and estimates uniform in $\boldsymbol{\theta}$.

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and study the operator family $A(\mathbf{k}) = A(t\boldsymbol{\theta}) =: A(t, \boldsymbol{\theta})$ by methods of the analytic perturbation theory with respect to the one-dimentional parameter t. But we have to make our constructions and estimates uniform in $\boldsymbol{\theta}$.
By the Rellich–Kato theorem, for $t \leqslant t^0$ there exist real-analytic branches of eigenvalues $\lambda_l(t, \boldsymbol{\theta})$ and eigenvectors $\varphi_l(t, \boldsymbol{\theta})$ of the operator $A(t, \boldsymbol{\theta})$, $l = 1, \dots, n$. We have

$$A(t, \boldsymbol{\theta})\varphi_l(t, \boldsymbol{\theta}) = \lambda_l(t, \boldsymbol{\theta})\varphi_l(t, \boldsymbol{\theta}), \quad l = 1, \dots, n,$$

and $\{\varphi_l(t, \boldsymbol{\theta})\}$ form an orthonormal basis in the eigenspace of $A(t, \boldsymbol{\theta})$ corresponding to $[0, \delta]$.

Analytic perturbation theory

Then we have the following power series expansions

$$\lambda_l(t,\boldsymbol{\theta}) = \gamma_l(\boldsymbol{\theta})t^2 + \mu_l(\boldsymbol{\theta})t^3 + \dots, \quad l = 1,\dots,n,$$
$$\varphi_l(t,\boldsymbol{\theta}) = \omega_l(\boldsymbol{\theta}) + t\varphi_l^{(1)}(\boldsymbol{\theta}) + \dots, \quad l = 1,\dots,n.$$

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Here $\gamma_l(\boldsymbol{\theta}) \geqslant c_* > 0$ and $\mu_l(\boldsymbol{\theta}) \in \mathbb{R}$. The "embryos" $\omega_1(\boldsymbol{\theta}),\ldots,\omega_n(\boldsymbol{\theta})$ form an orthonormal basis in $\mathfrak{N}$.

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The operator $S(\boldsymbol{\theta}) = b(\boldsymbol{\theta})^* g^0 b(\boldsymbol{\theta})$ is called the *spectral germ* of the operator family $A(t,\boldsymbol{\theta})$ at $t = 0$.

Proposition 1 [M. Birman and T. Suslina, 2003]

$$S(\boldsymbol{\theta})\omega_l(\boldsymbol{\theta}) = \gamma_l(\boldsymbol{\theta})\omega_l(\boldsymbol{\theta}), \quad l = 1,\dots,n.$$

Threshold approximations

We need the so called threshold approximations for the spectral projection $F(t, \boldsymbol{\theta})$ of the operator $A(t, \boldsymbol{\theta})$ corresponding to $[0, \delta]$ and for the operator $A(t, \boldsymbol{\theta})F(t, \boldsymbol{\theta})$.

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Proposition 2 [M. Birman and T. Suslina, 2003]

Let $S(\boldsymbol{\theta}) = b(\boldsymbol{\theta})^* g^0 b(\boldsymbol{\theta})$. *For* $t \leqslant t^0$ *we have*

$$\|F(t, \boldsymbol{\theta}) - P\|_{L_2(\Omega) \to L_2(\Omega)} \leqslant C_1 t, \tag{9}$$

$$\|A(t, \boldsymbol{\theta})F(t, \boldsymbol{\theta}) - t^2 S(\boldsymbol{\theta})P\|_{L_2(\Omega) \to L_2(\Omega)} \leqslant C_2 t^3 \tag{10}$$

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Proposition 3 [M. Birman and T. Suslina, 2008]

$$\|A(t,\boldsymbol{\theta})^{1/2}F(t,\boldsymbol{\theta}) - (t^2 S(\boldsymbol{\theta}))^{1/2}P\|_{L_2(\Omega)\to L_2(\Omega)} \leqslant C_3 t^2 \tag{11}$$

Proof of estimate (8) with $s = 2$

We have

$$\left(e^{-i\tau A(t,\boldsymbol{\theta})^{1/2}} - e^{-i\tau(t^2 S(\boldsymbol{\theta}))^{1/2}}\right) P$$
$$= e^{-i\tau A(t,\boldsymbol{\theta})^{1/2}}(P - F(t,\boldsymbol{\theta}))P - (P - F(t,\boldsymbol{\theta}))e^{-i\tau(t^2 S(\boldsymbol{\theta}))^{1/2}} P -$$
$$- i \int_0^\tau e^{i(\tilde{\tau}-\tau)A(t,\boldsymbol{\theta})^{1/2}} \left(A(t,\boldsymbol{\theta})^{1/2} F(t,\boldsymbol{\theta}) - (t^2 S(\boldsymbol{\theta}))^{1/2} P\right) e^{-i\tilde{\tau}(t^2 S(\boldsymbol{\theta}))^{1/2}} P \, d\tilde{\tau}.$$

Using estimates (9) and (11), we obtain

$$\left\| \left(e^{-i\tau A(t,\boldsymbol{\theta})^{1/2}} - e^{-i\tau(t^2 S(\boldsymbol{\theta}))^{1/2}}\right) P \right\| \leqslant 2C_1 t + C_3 |\tau| t^2, \quad t \leqslant t^0.$$

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$$\leqslant (2C_1 t + C_3 \varepsilon^{-1} |\tau| t^2) \varepsilon^2 (t^2+\varepsilon^2)^{-1} \leqslant (C_1 + C_3|\tau|)\varepsilon, \quad t \leqslant t^0.$$

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This completes the proof. We arrive at the following result.

Theorem 1 [M. Birman and T. Suslina, 2008]

$$\| \cos(\tau A_\varepsilon^{1/2}) - \cos(\tau (A^0)^{1/2}) \|_{H^2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leqslant (C_1 + C_3|\tau|)\varepsilon.$$

More accurate threshold approximations

For more subtle considerations, we need more accurate approximation for $A(t,\boldsymbol{\theta})F(t,\boldsymbol{\theta})$:

Proposition 4 [M. Birman and T. Suslina, 2005]

For $t \leqslant t^0$ we have

$$A(t,\boldsymbol{\theta})F(t,\boldsymbol{\theta}) = t^2 S(\boldsymbol{\theta})P + t^3 K(\boldsymbol{\theta}) + \Psi(t,\boldsymbol{\theta}). \tag{12}$$

The remainder term $\Psi(t,\boldsymbol{\theta})$ satisfies

$$\|\Psi(t,\boldsymbol{\theta})\|_{L_2(\Omega)\to L_2(\Omega)} \leqslant C_4 t^4, \quad t \leqslant t^0. \tag{13}$$

The operator $K(\boldsymbol{\theta})$ is described in the invariant terms, as well as in terms of the coefficients of power series expansions for the eigenvalues and the eigenvectors of $A(t,\boldsymbol{\theta})$.

More accurate threshold approximations

For our problem, only the block of $K(\boldsymbol{\theta})$ in the subspace $\mathfrak{N}$ is important. We have the following invariant representation:

$$N(\boldsymbol{\theta}) := PK(\boldsymbol{\theta})P = b(\boldsymbol{\theta})^* L(\boldsymbol{\theta}) b(\boldsymbol{\theta}) P,$$

$$L(\boldsymbol{\theta}) := |\Omega|^{-1} \int_\Omega \left(\Lambda(\mathbf{x})^* b(\boldsymbol{\theta})^* \widetilde{g}(\mathbf{x}) + \widetilde{g}(\mathbf{x})^* b(\boldsymbol{\theta}) \Lambda(\mathbf{x}) \right) d\mathbf{x}.$$

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In terms of the coefficients,

$$N(\boldsymbol{\theta}) = N_0(\boldsymbol{\theta}) + N_*(\boldsymbol{\theta}),$$

$$N_0(\boldsymbol{\theta}) = \sum_{l=1}^{n} \mu_l(\boldsymbol{\theta})(\cdot, \omega_l(\boldsymbol{\theta}))\omega_l(\boldsymbol{\theta}),$$

$$N_*(\boldsymbol{\theta}) = \sum_{l=1}^{n} \gamma_l(\boldsymbol{\theta}) \left((\cdot, \widetilde{\omega}_l(\boldsymbol{\theta}))\omega_l(\boldsymbol{\theta}) + (\cdot, \omega_l(\boldsymbol{\theta}))\widetilde{\omega}_l(\boldsymbol{\theta}) \right),$$

Here $\widetilde{\omega}_l(\boldsymbol{\theta}) \in \mathfrak{N}$ are defined by $\widetilde{\omega}_l(\boldsymbol{\theta}) := P\varphi_l^{(1)}(\boldsymbol{\theta})$.

More accurate threshold approximations

Proposition 5 [M. Dorodnyi and T. Suslina, 2016]

For $t \leqslant t^0$ *we have*

$$A(t, \boldsymbol{\theta})^{1/2} F(t, \boldsymbol{\theta}) = t\, S(\boldsymbol{\theta})^{1/2} P + t^2 G(\boldsymbol{\theta}) + \Phi(t, \boldsymbol{\theta}).$$

The remainder term $\Phi(t, \boldsymbol{\theta})$ *satisfies*

$$\|\Phi(t, \boldsymbol{\theta})\|_{L_2(\Omega) \to L_2(\Omega)} \leqslant C_5 t^3, \quad t \leqslant t^0.$$

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Only the block of $G(\boldsymbol{\theta})$ in the subspace $\mathfrak{N}$ is needed:

$$PG(\boldsymbol{\theta})P = \frac{1}{2}N_0(\boldsymbol{\theta})S(\boldsymbol{\theta})^{-1/2}P + S(\boldsymbol{\theta})^{-1/2}N_*(\boldsymbol{\theta})P + \\ + N_*(\boldsymbol{\theta})S(\boldsymbol{\theta})^{-1/2}P + \mathcal{I}_*(\boldsymbol{\theta})P,$$

where

$$\mathcal{I}_*(\boldsymbol{\theta}) = -\frac{1}{\pi}\int_0^\infty s^{-1/2}\big(\Xi(t,s)N_*(\boldsymbol{\theta}) + N_*(\boldsymbol{\theta})\Xi(t,s) - s\Xi(t,s)N_*(\boldsymbol{\theta})\Xi(t,s)\big)\,ds$$

and $\Xi(t,s) = (t^2 S(\boldsymbol{\theta}) + sI)^{-1}P$.

Improvement of the result under additional assumptions

Theorem 2 [M. Dorodnyi and T. Suslina, 2016]

Suppose that $N(\boldsymbol{\theta}) = 0$ *for any* $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$. *Then*

$$\| \cos(\tau A_\varepsilon^{1/2}) - \cos(\tau (A^0)^{1/2}) \|_{H^{3/2}(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leqslant (\widetilde{C}_1 + \widetilde{C}_2 |\tau|)\varepsilon. \qquad (14)$$

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The proof is similar to that of Theorem 1. We use more accurate threshold approximations.

Remark. If $A_\varepsilon = -\operatorname{div} g^\varepsilon(\mathbf{x}) \nabla$, where $g(\mathbf{x})$ is a symmetric matrix with real entries, then $N(\boldsymbol{\theta}) = 0$ for any $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$. Hence, (14) is true.

Improvement of the result under additional assumptions

Theorem 3 [M. Dorodnyi and T. Suslina, 2016]

Suppose that $N_0(\boldsymbol{\theta}) = 0$ *for any* $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$ *(this is equivalent to the assumption that* $\mu_l(\boldsymbol{\theta}) = 0$ *for all* $l = 1, \ldots, n$ *and any* $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$*). Suppose that the number* p *of different eigenvalues of* $S(\boldsymbol{\theta})$ *does not depend on* $\boldsymbol{\theta}$*. Then*

$$\| \cos(\tau A_\varepsilon^{1/2}) - \cos(\tau (A^0)^{1/2}) \|_{H^{3/2}(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leqslant (\check{C}_1 + \check{C}_2 |\tau|)\varepsilon. \qquad (15)$$

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Corollary

Suppose that A_ε has real-valued coefficients. Suppose also that all eigenvalues of the germ $S(\boldsymbol{\theta})$ are simple for any $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$. Then estimate (15) is valid.

Sharpness of [BSu, 2008] estimate in the general case

Finally, we confirm that Theorem 1 is sharp in the following sense.

Theorem 4 [M. Dorodnyi and T. Suslina, 2016]

Suppose that $N_0(\boldsymbol{\theta}_0) \neq 0$ *for some* $\boldsymbol{\theta}_0 \in \mathbb{S}^{d-1}$ *(it means that* $\mu_l(\boldsymbol{\theta}_0) \neq 0$ *for some* l*). Let* $\tau \neq 0$ *and* $s < 2$*. Then there does not exist a constant* $C(\tau) > 0$ *such that the estimate*

$$\| \cos(\tau A_\varepsilon^{1/2}) - \cos(\tau (A^0)^{1/2}) \|_{H^s(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leqslant C(\tau)\varepsilon$$

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holds for all sufficiently small $\varepsilon > 0$.

There are concrete examples of operators A_ε satisfying the assumptions of Theorem 4. One example is of the form $-\mathrm{div}\, g^\varepsilon(\mathbf{x})\nabla$, where $g(\mathbf{x})$ is Hermitian matrix with complex entries. Another example is the matrix operator with real-valued coefficients (the operator of elasticity theory in the cases of anisotropic and isotropic media, $d = 2$).

Generalization. Applications

Similar results are obtained for more general operators of the form

$$\widetilde{A}_\varepsilon = (f^\varepsilon(\mathbf{x}))^* b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D}) f^\varepsilon(\mathbf{x}),$$

where $f(\mathbf{x})$ is a Γ-periodic $(n \times n)$-matrix-valued function such that $f, f^{-1} \in L_\infty$.

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We apply the results to the following equations:

- The acoustics equation,
- The system of elasticity theory.

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