On the second mixed moment of the characteristic polynomials of sparse hermitian random matrices

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Outline

1. Basic concepts of the random matrix theory
2. The mixed moments of the characteristic polynomials
3. The ensemble of the sparse random matrices
4. The Grassmann variables method
5. Formulations of the theorems

Random matrices

Definition

The sequence M_n of the $n \times n$ matrices, which entries are random variables, is called a random matrix ensemble.

Examples

- Wigner ensemble

$$M^T = M \text{ or } M^* = M$$
$$M_{jk} = n^{-1/2} w_{jk},$$
$$\{w_{jk}\} - \text{i.i.d.}, \quad E\{w_{jk}\} = 0, \quad E\{|w_{jk}|^2\} = 1$$

- The adjacency matrices of random graphs

$$M_{jk} = \begin{cases} 1 \text{ with probability } \frac{p_n}{n}; \\ 0 \text{ with probability } 1 - \frac{p_n}{n}. \end{cases}$$

Global regime

Normalized counting measure of eigenvalues (NCM) and linear eigenvalue statistics

$$N_n(\Delta) = \frac{1}{n} \sum_{j=1}^{n} \mathbb{1}_\Delta(\lambda_j^{(n)}), \quad \mathcal{N}_n[\varphi] = \sum_{j=1}^{n} \varphi(\lambda_j^{(n)}) = \operatorname{tr} \varphi(M)$$

Questions

1. $N_n(\Delta) \xrightarrow{w} N(\Delta)$
2. Central Limit Theorem for linear eigenvalue statistics

$\sigma = \operatorname{supp} N$ is called the spectrum.

For the Wigner ensemble

$$N(\Delta) = \int_\Delta \rho_{sc}(\lambda) d\lambda, \quad \rho_{sc}(\lambda) = \frac{1}{2\pi} \sqrt{4 - \lambda^2} \cdot \mathbb{1}_{[-2,2]}, \quad \sigma = [-2, 2].$$

Local regime

The spectral correlation functions are

$$p_k^{(n)}(\lambda_1, \ldots, \lambda_k) = \int p_n(\lambda_1, \ldots, \lambda_n) d\lambda_{k+1} \ldots d\lambda_n.$$

where $p_n(\lambda_1, \ldots, \lambda_n)$ is the joint probability density of the eigenvalues.

Dyson universality conjecture

$$\lim_{n\to\infty} (\rho_n(\lambda_0))^{-k} p_k^{(n)}(\lambda_0 + x_1/n\rho_n(\lambda_0), \ldots, \lambda_0 + x_k/n\rho_n(\lambda_0)) = \det \left\{ \frac{\sin \pi(x_i - x_j)}{\pi(x_i - x_j)} \right\}_{i,j=1}^{k}$$

where $\rho_n(\lambda) = p_1^{(n)}(\lambda)$.

The mixed moments of the characteristic polynomials

Let M_n be some ensemble of random matrices. Consider the second mixed moment or the correlation function of characteristic polynomials

$$F_2(\Lambda) = E\{\det(M_n - \lambda_1)\det(M_n - \lambda_2)\}, \quad \Lambda = \operatorname{diag}\{\lambda_j\}_{j=1}^2$$

$$a)\ \lambda_j = \lambda_0 + \frac{x_j}{n}, \qquad b)\ \lambda_j = \lambda_0 + \frac{x_j}{n^{2/3}}$$

$$F_2(\Lambda) \xrightarrow[n\to\infty]{} ?$$

Some results

- Keating, Snaith (2000)
- Brezin, Hikami (2000, 2001)
- Strahov, Fyodorov (2002, 2003); Fyodorov, Khoruzhenko (2006)
- Götze, Kösters (2008, 2009)
- T. Shcherbina (2011, 2013, 2014, 2015)

Sparse random matrices

Ensemble of the sparse hermitian random matrices

$$M_n = (d_{jk} w_{jk})_{j,k=1}^n,$$

where

$$d_{jk} = p^{-1/2} \begin{cases} 1 \text{ with probability } \frac{p}{n}; \\ 0 \text{ with probability } 1 - \frac{p}{n}. \end{cases}$$

and $\Re w_{jk}$, $\Im w_{jk}$, w_{ll} are independent Gaussian random variables with zero mean such that

$$E\{|w_{jk}|^2\} = 1.$$

The ensemble is studied in two regimes

1. $\lim\limits_{n\to\infty} p < \infty$
2. $\lim\limits_{n\to\infty} p = \infty$

The results on the ensemble of sparse random matrices

Global regime

- For $p \to \infty$ the normalized counting measure is the same as for the Wigner ensemble.
  - Rodgers, Bray (1988) on physical level of rigour;
  - Khorunzhy, Khoruzhenko, Pastur and M. Shcherbina (1992).
- For finite p the convergence of the normalized counting measure was proven by
  - Rodgers, De Dominicis (1990) on physical level of rigour;
  - Bauer, Golinelli (2001) for $w_{jk} = 1$;
  - Khorunzhy, M. Shcherbina, Vengerovsky (2004) in general case.
- Central Limit Theorem for linear eigenvalue statistics was proven by M. Shcherbina, Tirozzi for finite p (2010) and for $p \to \infty$ (2012).

The results on the ensenble of sparse random matrices

Local regime

- In the papers by Erdős, Knowles, Yau, Yin (2012) and Huang, Landon, Yau (2015) it was rigorously proved that for $p \gg n^{\varepsilon}$ the spectral correlation functions converge in weak sense to that for Wigner ensemble.
- The conjecture of existing of the critical value $p_c > 1$ at which the correlation of eigenvalues is changed.
  - Evangelou, Economou (1992);
  - Fyodorov, Mirlin (1991, on the physical level of rigour).

Grassmann variables

Let $\{\psi_j, \overline{\psi}_j\}_{j=1}^n$ be a set of anticommuting variables, i.e.

$$\psi_j\psi_k + \psi_k\psi_j = \overline{\psi}_j\psi_k + \psi_k\overline{\psi}_j = \overline{\psi}_j\overline{\psi}_k + \overline{\psi}_k\overline{\psi}_j = 0.$$

In particular, $\psi_j^2 = \overline{\psi}_k^2 = 0$. The set generates a graded algebra $\mathcal{A}$ of polynomials of $\{\psi_j, \overline{\psi}_j\}$, which is called the Grassmann algebra.
For an analytical function f it's domain can be extended to Grassmann algebra by following.

$$f(\chi + z_0) = \sum_{j=0}^{\infty} \frac{f^{(j)}(z_0)}{j!}\chi^j,$$

where χ is a polynomial of $\{\psi_j, \overline{\psi}_j\}$ with zero free term.

Integration over the Grassmann variables

The integral over the Grassmann variables is a linear functional, defined on the basis by the relations

$$\int d\psi_j = \int d\overline{\psi}_k = 0, \qquad \int \psi_j d\psi_j = \int \overline{\psi}_k d\overline{\psi}_k = 1.$$

A multiple integral is defined to be the repeated integral. Moreover "differentials" $\{d\psi_j, d\overline{\psi}_j\}_{j=1}^n$ anticommute with each other and with $\{\psi_j, \overline{\psi}_j\}_{j=1}^n$.
For example, for a function f

$$f(\psi_1, \dots, \psi_n) = a_0 + \sum_{j=1}^n a_j \psi_j + \dots + a_{1,\dots,n} \prod_{j=1}^n \psi_j$$

we have by definition

$$\int f(\psi_1, \dots, \psi_n) d\psi_n \dots d\psi_1 = a_{1,\dots,n}.$$

Integration over the Grassmann variables

Let A be a positive definite $n \times n$ matrix. The following Gaussian integral is well-known

$$\frac{1}{\pi^n}\int \exp\left\{-\sum_{j,k=1}^{n} \bar{z}_j A_{jk} z_k\right\}\prod_{j=1}^{n} d\Re z_j d\Im z_j = \frac{1}{\det A}.$$

The important analogue of this formula in Grassmann variables theory

$$\int \exp\left\{-\sum_{j,k=1}^{n} \overline{\psi}_j A_{jk} \psi_k\right\}\prod_{j=1}^{n} d\overline{\psi}_j d\psi_j = \det A \qquad (1)$$

is valid for any matrix A.

If A is a hermitian matrix with i.i.d. Gaussian entries then the l.h.s. of (1) can be easily averaged

$$E\{\det A\} = \int \exp\left\{\sum_{j<k} \overline{\psi}_j \psi_k \overline{\psi}_k \psi_j\right\}\prod_{j=1}^{n} d\overline{\psi}_j d\psi_j$$

The derivation of the integral representation

$$F_2(\Lambda) = \mathrm{E}\left\{\det(M_n - \lambda_1)\det(M_n - \lambda_2)\right\}$$
$$= C\int \exp\left\{\Phi\left((\overline{\psi}_1,\psi_1),(\overline{\psi}_1,\psi_2),(\overline{\psi}_2,\psi_1),(\overline{\psi}_2,\psi_2)\right)\right\} d\Psi,$$

where Φ is an even polynomial of the 4th degree. Using the Hubbard-Stratonovich transformation

$$e^{y^2} = \frac{a}{\sqrt{\pi}}\int e^{2axy - a^2x^2}dx,$$
$$e^{yt} = \frac{a^2}{\pi}\int e^{ay(u+iv)+at(u-iv)-a^2u^2-a^2v^2}dudv$$

we return to the usual integral representation

$$F_2(\Lambda) = C\iint\prod_j \exp\{-\frac{1}{2}\,\mathrm{tr}\,Q^2 + g(\overline{\psi}_{j1},\psi_{j1},\overline{\psi}_{j2},\psi_{j2},Q)\}dQd\Psi = C\int\ldots dQ,$$

where Q is 2×2 hermitian matrix.

Final integral representation

$$F_2(\Lambda) = C_n(X)\frac{ie^{\lambda_0(x_1+x_2)}}{x_1 - x_2}\int\limits_{\mathbb{R}^3}(t_1 - t_2)\exp\Big\{-i\sum_{j=1}^{2} x_j t_j\Big\}e^{nf(t_1,t_2,s)}dt_1dt_2ds,$$

where

$$f(t_1, t_2, s) = \log\left(s\sqrt{\frac{2(n-p)}{np}} - t_1 t_2\right) - \frac{1}{2}\left(\sum_{j=1}^{2}(t_j + i\lambda_0)^2 + s^2\right).$$

The second order correlation function

Theorem 1 [A.:16 (published in JSP)]

Consider the normalized second order correlation function

$$D_2(\Lambda) = \frac{F_2(\Lambda)}{\sqrt{F_2(\lambda_1 I) F_2(\lambda_2 I)}}.$$

Then we have for finite p

(i) for $p > 2$

$$\lim_{n\to\infty} D_2(\Lambda) = \begin{cases} \frac{\sin((x_1 - x_2)\sqrt{\lambda_*^2 - \lambda_0^2}/2)}{(x_1 - x_2)\sqrt{\lambda_*^2 - \lambda_0^2}/2}, & \text{if } |\lambda_0| < \lambda_*, \\ 1, & \text{if } |\lambda_0| \geq \lambda_*; \end{cases}$$

with $\lambda_* = \sqrt{\left(4 - \frac{8}{p}\right)_+}$.

(ii) for $p \leq 2$

$$\lim_{n\to\infty} D_2(\Lambda) = 1,$$

where $\Lambda = \text{diag}\{\lambda_1, \lambda_2\} = \text{diag}\left\{\lambda_0 + \frac{x_1}{n}, \lambda_0 + \frac{x_2}{n}\right\}$, $\lambda_0, x_1, x_2 \in \mathbb{R}$.

The second order correlation function at the edge of the spectrum

Theorem 2 [A.:16 (published in JSP)]

Let $p \to \infty$ and $\lambda_0 = 2$. Then

(i) for $\frac{n^{2/3}}{p} \to \infty$

$$\lim_{n\to\infty} D_2\left(2I + X/n^{2/3}\right) = 1;$$

(ii) for $\frac{n^{2/3}}{p} \to c$

$$\lim_{n\to\infty} D_2\left(2I + X/n^{2/3}\right) = \frac{\mathbb{A}(x_1 + 2c, x_2 + 2c)}{\sqrt{\mathbb{A}(x_1 + 2c, x_1 + 2c)\mathbb{A}(x_2 + 2c, x_2 + 2c)}},$$

where $X = \mathrm{diag}\{x_1, x_2\}$, $\mathbb{A}(x, y) = \frac{Ai(x)Ai'(y) - Ai'(x)Ai(y)}{x - y}$ and $Ai(x)$ is Airy function.

The correlation functions of higher order

Theorem 3 [A.:16 (published in JSP)]

Let $p \to \infty$, $\lambda_0 \in (-2, 2)$. Then

$$\lim_{n\to\infty} \frac{F_{2m}(\Lambda)}{\left(\prod_{j=1}^{2m} F_{2m}(\lambda_j I)\right)^{\frac{1}{2m}}} = \frac{\hat{S}_{2m}(X)}{\hat{S}_{2m}(I)},$$

where

$$\hat{S}_{2m}(X) = \frac{\det\left\{\dfrac{\sin(\pi\rho_{sc}(\lambda_0)(x_j - x_{m+k}))}{\pi\rho_{sc}(\lambda_0)(x_j - x_{m+k})}\right\}_{j,k=1}^{m}}{\Delta(x_1,\dots,x_m)\Delta(x_{m+1},\dots,x_{2m})}.$$